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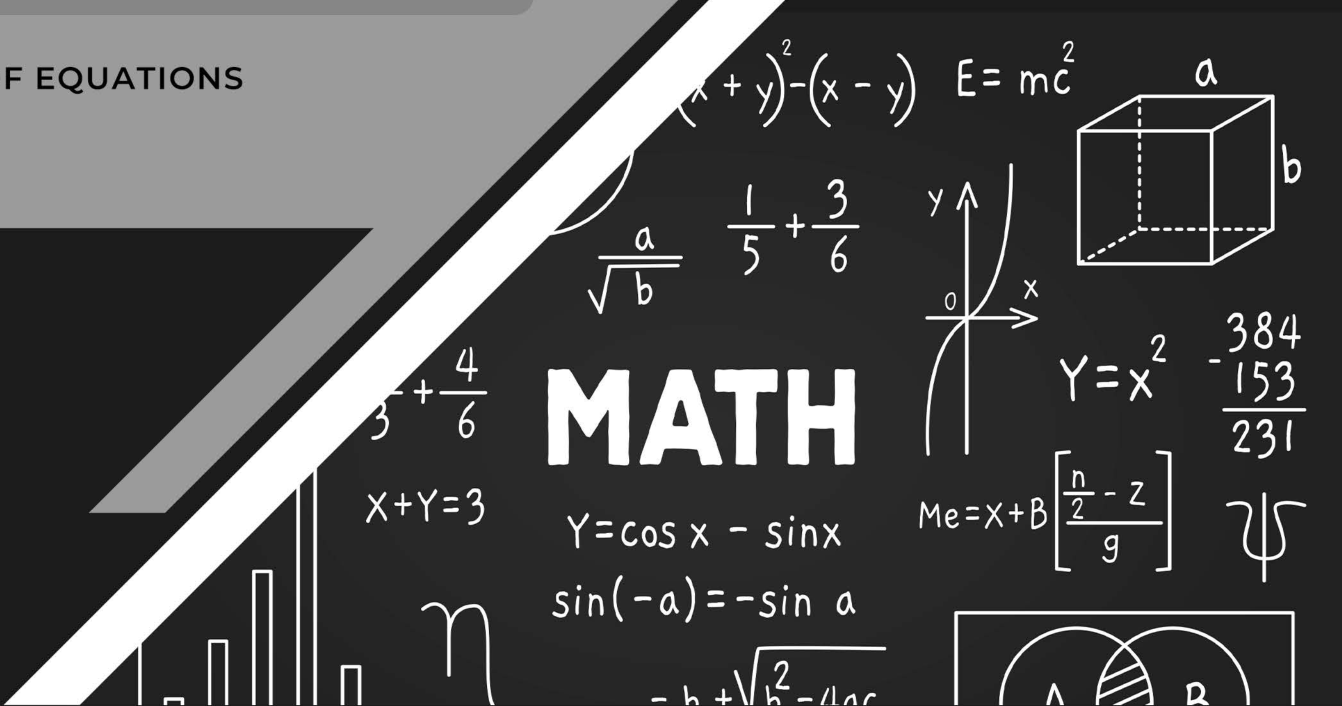
MATHEMATICS

CLASS XI + XII

THEORY OF EQUATIONS



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Theory of Equations

■ POLYNOMIALS

Some Definitions

Real Polynomial

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is called a real polynomial of real variable x with real coefficients.

Complex Polynomial

If $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_n x^n$ is called a complex polynomial or a polynomial of complex coefficients.

Rational Expression or Rational Function

An expression of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x, is called a rational expression.

In particular, when $Q(x)$ is a non-zero constant, $\frac{P(x)}{Q(x)}$ reduces to a polynomial. Thus, every polynomial is a rational expression but the converse is not true.

Few examples of rational expressions are:

(i) $\frac{x^2 - 5x + 4}{x - 2}$ (ii) $x^2 - 5x + 4$ (iii) $\frac{1}{x-2}$ (iv) $x + \frac{1}{x}$ i.e., $\frac{x^2+1}{x}$

Degree of a Polynomial

A polynomial $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, real or complex, is a polynomial of degree n, if $a_n \neq 0$. The polynomials $2x^3 - 7x^2 + x + 5$ and $(3 - 2i)x^2 - ix + 5$ are polynomials of degree 3 and 2, respectively. A polynomial of second degree is generally called a quadratic polynomial and polynomials of degree 3 and 4 are known as cubic and biquadratic polynomials.

Polynomial Equation

If $f(x)$ is a polynomial, then $f(x) = 0$ is called a polynomial equation. The general form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$. Here, x is the variable and a, b, c are coefficients which can be real or imaginary.

Roots of an Equation

The values of the variable satisfying a given equation are called its roots. Thus, $x = \alpha$, is a root of the equation $f(x) = 0$, if $f(\alpha) = 0$.

For example, $x = 1$ is a root of the equation $x^3 - 6x^2 + 11x - 6 = 0$, because $1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0$.

Solution Set

The set of all roots of an equation, in a given domain, is called the solution set of the equation. For example, the set {1, 2, 3} is the solution set of the equation $x^3 - 6x^2 + 11x - 6 = 0$. Solving an equation means finding its solution set. In other words, solving an equation is the process of obtaining its all roots.

REMAINDER AND FACTOR THEOREMS

Remainder Theorem

The Remainder Theorem states that if a polynomial $f(x)$ is divided by a linear function $x - k$, the remainder is $f(k)$.

Proof:

In any division,

Dividend = Divisor × Quotient + Remainder

Let $Q(x)$ be the quotient and R be remainder.

$\Rightarrow \quad f(x) = (x - k)\,Q(x) + R$

$\Rightarrow \quad f(k) = (k - k)\,Q(x) + R = 0 + R = R$

Note:

☑ The degree of the remainder is always one less than the degree of the divisor.

Factor Theorem

Factor Theorem is a special case of Remainder Theorem.

$f(x) = (x - k)\,Q(x) + R$

or $\quad f(x) = (x - k)\,Q(x) + f(k)$

When $\quad f(k) = 0$, then $f(x) = (x - k)\,Q(x)$. Therefore, $f(x)$ is exactly divisible by $x - k$.

Illustration 1 Given that the expression $2x^3 + 3px^2 - 4x + p$ has a remainder of 5 when divided by $x + 2$. Find the value of p.

Solution

Let $f(x) = 2x^3 + 3px^2 - 4x + p$

According to the question, we have

$f(-2) = 2(-2)^3 + 3(-2)^2 p - 4(-2) + p = 5$

$\Rightarrow \quad 13p - 8 = 5 \Rightarrow p = 1$

Illustration 2 When an unknown polynomial is divided by $(x - 1)$ and $(x - 2)$, we obtain the remainders 2 and 1, respectively. Find the remainder resulting from the division of this polynomial by $(x - 1)(x - 2)$.

Solution

Let $P(x)$ be the unknown polynomial and let $q(x)$ be the quotient and $r(x) = ax + b$ be the remainder resulting from the division of that polynomial by $(x - 1)(x - 2)$.

$\Rightarrow \quad P(x) = (x - 1)(x - 2)\,q(x) + ax + b$

According to the question, $P(1) = 2$

$\Rightarrow \quad a + b = 2$

Also, $\quad P(2) = 1$

$\Rightarrow \quad 2a + b = 1$

So, $\quad a = -1, b = 3$

$\therefore \quad$ Required remainder $= (3 - x)$

Illustration 3 Given that $x^2 + x - 6$ is a factor of $2x^4 + x^3 - ax^2 + bx + a + b - 1$. Find the values of a and b.

Solution

$x^2 + x - 6 = (x + 3)(x - 2)$

Let $f(x) = 2x^4 + x^3 - ax^2 + bx + a + b - 1$

$f(-3) = 2(-3)^4 + (-3)^3 - a(-3)^2 - 3b + a + b - 1 = 0$

$\Rightarrow$ $134 - 8a - 2b = 0$

$\Rightarrow$ $4a + b = 67$ ...(1)

$\Rightarrow$ $f(2) = 2(2)^4 + 2^3 - a(2)^2 + 2b + a + b - 1 = 0$

$\Rightarrow$ $39 - 3a + 3b = 0$

$\Rightarrow$ $a - b = 13$ ...(2)

From (1) and (2), $a = 16$, $b = 3$

Illustration 4 If c and d are the roots of the equation $(x - a)(x - b) - k = 0$, then find the equation having roots a and b.

Solution

Since c and d are the roots of the equation $(x - a)(x - b) - k = 0$, we have

$(x - a)(x - b) - k = (x - c)(x - d)$

$\Rightarrow$ $(x - a)(x - b) = (x - c)(x - d) + k$

$\Rightarrow$ $(x - c)(x - d) + k = (x - a)(x - b)$

Clearly, a and b are roots of the equation $(x - a)(x - b) = 0$.

So, a and b are roots of $(x - c)(x - d) + k = 0$.

IDENTITY

An equation which is true for every value of the variable is called an identity equation.

Examples of identity equation are: $5(a - 3) = 5a - 15$, $(a + b)^2 = a^2 + 2ab + b^2$

An inequality which is true for every values of the variable is called an identity inequality. For example, the inequality $a^2 \geq 0$ is true for every value of a.

Consider the equation $ax^2 + bx + c = 0$. We know that quadratic equation can have two real distinct roots or we can say that $ax^2 + bx + c$ can be zero for two real distinct values of x, say α and β.

Now, suppose it is given that $ax^2 + bx + c = 0$ is get satisfied by three values of x, say α, β and γ (where $\alpha \neq \beta \neq \gamma$)

$\therefore$ $a\alpha^2 + b\alpha + c = 0$ ...(i)

$a\beta^2 + b\beta + c = 0$ ...(ii)

$a\gamma^2 + b\gamma + c = 0$ ...(iii)

From (i) – (ii), $a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$ or $a(\alpha + \beta) + b = 0$ ...(iv)

From (ii) – (iii), $a(\beta^2 - \gamma^2) + b(\beta - \gamma) = 0$ or $a(\beta + \gamma) + b = 0$ ...(v)

From (iv) – (v), $a(\alpha - \gamma) = 0$

$\Rightarrow$ $a = 0$ (as $\alpha \neq \gamma$)

$\Rightarrow$ $b = 0$ (from (iv))

$\Rightarrow$ $c = 0$ (from (i))

Thus, $a = b = c = 0$. Hence, $ax^2 + bx + c = 0$ becomes $0x^2 + 0x + 0 = 0$. Now, this is satisfied by not only α, β and γ but by all real numbers. Thus, identity is true for all real numbers. Similarly, if cubic equation $ax^3 + bx^2 + cx + d = 0$ is satisfied by four values α, β, γ and δ, then $a = b = c = d = 0$.

In general, any polynomial equation of degree n is an identity if it is satisfied by more than n values of x. Then all the coefficients including constant term are zero.

Illustration 5 If $(a^2-1)x^2+(a-1)x+a^2-4a+3=0$ is an identity in x, then find the value of a.

Solution

The given relation is satisfied for all real values of x then all the coefficients must be zero.

Then,

$$\left.\begin{aligned} a^2-1=0 &\Rightarrow a=\pm 1 \\ a-1=0 &\Rightarrow a=1 \\ a^2-4a+3=0 &\Rightarrow a=1,3 \end{aligned}\right\}$$

So, common value of a is 1.

SOLVING EQUATIONS

Key Points in Solving Equations

Domain of Equation

It is a set of values of independent variable x for which each function used in the equation is defined i.e., it takes up finite real values. In other words, the final solution obtained while solving any equation must satisfy the domain of the expression of the parent equation.

For example, equation $\dfrac{x^2-2x-3}{x+1}=0$ is solvable over $R-\{-1\}$.

Now, $\dfrac{x^2-2x-3}{x+1}=0 \Rightarrow x^2-2x-3=0$ or $(x-3)(x+1)=0 \Rightarrow x=3$ (as $x \in R-\{-1\}$)

Extraneous Roots

While simplifying the equation, the domain of the equation may expand and give the extraneous roots.

For example, consider the equation $\sqrt{x}=x-2$.

$\Rightarrow \quad x=(x-2)^2$ [On squaring both sides]

$\Rightarrow \quad x^2-5x+4=0 \Rightarrow (x-1)(x-4)=0 \Rightarrow x=1,4$

We observe that $x=4$ satisfies the given equation but $x=1$ does not satisfy it.

Hence, $x=4$ is the only solution of the given equation.

The domain of original equation is $[2,\infty)$.

On squaring the equation, domain expands to R, which gives extra root $x=1$.

Loss of Root

Cancellation of common factors from both sides of an equation lead to loss of root.

For example, consider an equation $x^2-2x=x-2$.

$\Rightarrow \quad x(x-2)=x-2 \Rightarrow x=1$

Here, we have cancelled factor $x-2$ which cause the loss of root, $x=2$

The correct way of solving the given equation is:

$$x^2-2x=x-2 \Rightarrow x^2-3x+2=0 \Rightarrow (x-1)(x-2)=0 \Rightarrow x=1 \text{ and } x=2.$$

Illustration 6 How many real roots does the equation $\sqrt{x-2}\,(x^2-4x+3)=0$ have?

Solution

Domain of the equation is $[2,\infty)$.

The roots of the equation are $x=2$ and $x=3$.

Illustration 7 Solve the equation $\sqrt{x+5}+\sqrt{x+21}=\sqrt{6x+40}$.

Solution

$$\sqrt{x+5}+\sqrt{x+21}=\sqrt{6x+40}$$

$\Rightarrow$ $\left(\sqrt{x+5}+\sqrt{x+21}\right)^2=6x+40 \Rightarrow (x+5)+(x+21)+2\sqrt{(x+5)(x+21)}=6x+40$

$\Rightarrow$ $\sqrt{(x+5)(x+21)}=2x+7 \Rightarrow (x+5)(x+21)=(2x+7)^2$

$\Rightarrow$ $3x^2+2x-56=0 \Rightarrow (3x+14)(x-4)=0 \Rightarrow x=4$ or $x=-14/3$

Clearly, $x=-14/3$ does not satisfy the given equation.

Hence, $x=4$ is the only root of the given equation.

Equations Reducible to Quadratic

Many polynomials and rational equations are not quadratic. But suitable substitution reduces the given equation to quadratic one. The following illustrations explain this.

Illustration 8 Solve the equation $(x^2+2)^2+8x^2=6x(x^2+2)$.

Solution

Given equation is $(x^2+2)^2+8x^2=6x(x^2+2)$

Dividing throughout by x^2, we get

$$\left(\frac{x^2+2}{x}\right)^2+8=6\left(\frac{x^2+2}{x}\right)$$

$\Rightarrow$ $t^2-6t+8=0$ where $t=\dfrac{x^2+2}{x}$

$\therefore$ $t=4, t=2$

If $\dfrac{x^2+2}{x}=4 \Rightarrow x=2\pm\sqrt{2}$ and if $\dfrac{x^2+2}{x}=2 \Rightarrow x=1\pm i$

Illustration 9 Solve the equation $(x^2-5x+7)^2-(x-2)(x-3)=1$.

Solution

We have

$(x^2-5x+7)^2-(x-2)(x-3)=1$

$\Rightarrow$ $(x^2-5x+7)^2-(x^2-5x+7)=0$

$\Rightarrow$ $y^2-y=0$, where $y=x^2-5x+7$

$\Rightarrow$ $y(y-1)=0$

$\Rightarrow$ $y=0, 1$

If $y=0$, then

$x^2-5x+7=0$

$\Rightarrow$ $x=\dfrac{5\pm\sqrt{25-28}}{2} \Rightarrow x=\dfrac{5\pm\sqrt{-3}}{2} \Rightarrow x=\dfrac{5\pm i\sqrt{3}}{2}$

If $y=1$, then

$x^2-5x+6=0 \Rightarrow (x-3)(x-2)=0 \Rightarrow x=3, 2$

Hence, the roots of the equation are 2, 3, $\dfrac{5+i\sqrt{3}}{2}$ and $\dfrac{5-i\sqrt{3}}{2}$.

Illustration 10 Solve the equation $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.

Solution

The given equation is $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.

In this equation, the coefficients of the terms equidistant from the two ends are equal. So, first divide both sides by x^2 and then regroup the terms containing same coefficient.

Dividing both sides of equation by x^2, we get

$$12x^2 - 56x + 89 - \frac{56}{x} + \frac{12}{x^2} = 0$$

$$\Rightarrow \quad 12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0 \Rightarrow 12\left[\left(x + \frac{1}{x}\right)^2 - 2\right] - 56\left(x + \frac{1}{x}\right) + 89 = 0$$

$$\Rightarrow \quad 12\left(x + \frac{1}{x}\right)^2 - 56\left(x + \frac{1}{x}\right) + 65 = 0 \Rightarrow 12y^2 - 56y + 65 = 0, \text{ where } y = x + \frac{1}{x}.$$

$$\Rightarrow \quad 12y^2 - 26y - 30y + 65 = 0 \Rightarrow (6y - 13)(2y - 5) = 0 \Rightarrow y = \frac{13}{6} \text{ or } y = \frac{5}{2}$$

If $\quad y = \frac{13}{6}$, then

$$x + \frac{1}{x} = \frac{13}{6} \Rightarrow 6x^2 - 13x + 6 = 0 \Rightarrow (3x - 2)(2x - 3) = 0 \Rightarrow x = 2/3, 3/2$$

If $\quad y = \frac{5}{2}$, then

$$x + \frac{1}{x} = \frac{5}{2} \Rightarrow 2x^2 - 5x + 2 = 0 \Rightarrow (x - 2)(2x - 1) = 0 \Rightarrow x = 2, 1/2$$

Hence, the roots of the given equation are 2, $\frac{1}{2}, \frac{2}{3}, \frac{3}{2}$.

Concept Application Exercise 1

1. Solve $4^x + 6^x = 9^x$.
2. Solve the following equations:
   (i) $x(x + 2)(x^2 - 1) = -1$ (ii) $\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1$
3. Solve for x: $\left(5 + 2\sqrt{6}\right)^{x^2 - 3} + \left(5 - 2\sqrt{6}\right)^{x^2 - 3} = 10$
4. Let $x = 1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 \ldots \infty}}}}$. Then the value of x is
   (1) $\frac{\sqrt{5}}{3}$ (2) $\frac{5}{\sqrt{3}}$ (3) $\sqrt{\frac{5}{3}}$ (4) $\frac{5}{3}$
5. Solve: $\frac{x^2 + 3x + 2}{x^2 - 6x - 7} = 0$
6. Determine the value of k for which $x + 2$ is a factor of $(x + 1)^7 + (2x + k)^3$.
7. Find the value of p for which $x + 1$ is a factor of $x^4 + (p - 3)x^3 - (3p - 5)x^2 + (2p - 9)x + 6$. Also, find the remaining factors for this value of p.
8. The value of k for which $(a + 2b)$, where $a, b \neq 0$, is a factor of $a^4 + 32b^4 + a^3b(k + 3)$ is
   (1) 3 (2) –1 (3) 2 (4) –2
9. If the expression $ax^4 + bx^3 - x^2 + 2x + 3$ has remainder $4x + 3$ when divided by $x^2 + x - 2$, then find the values of a and b.
10. Show that $\frac{(x + b)(x + c)}{(b - a)(c - a)} + \frac{(x + c)(x + a)}{(c - b)(a - b)} + \frac{(x + a)(x + b)}{(a - c)(b - c)} = 1$ is an identity.

NATURE OF ROOTS OF QUADRATIC EQUATION

Consider the quadratic equation $ax^2 + bx + c = 0$ ...(i)

Here, $a, b, c \in R$ and $a \neq 0$.

Roots of the equation are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Now, if we look at these roots, we observe that the nature of roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by D and is known as the discriminant of the quadratic equation (i). We also observe the following results:

- $D = 0$: Roots are equal $\alpha = \beta = -b/2a$
- $D \neq 0$: Roots are unequal
  - $a, b, c \in R$ and $D > 0$
    - $a, b, c \in Q$ and D is a perfect square $\Rightarrow$ Roots are rational
      - $a = 1, b, c \in Z$ and D is a perfect square $\Rightarrow$ Roots are integral
    - $a, b, c \in Q$ and D is not a perfect square $\Rightarrow$ Roots are irrational $\alpha = p + \sqrt{q}, \beta = p - \sqrt{q}$
  - $a, b, c \in R$ and $D < 0$: Roots are imaginary $\alpha = p + iq, \beta = p - iq$

Note:

- ☑ If $a, b, c \in Q$ and $b^2 - 4ac$ is positive but not a perfect square, then roots are irrational and they always occur in conjugate pair like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, if all a, b, c are not rational numbers and $b^2 - 4ac$ is positive but not a perfect square, then the roots may not occur in conjugate pairs. For example, the roots of the equation $x^2 - (5 + \sqrt{2})x + 5\sqrt{2} = 0$ are 5 and $\sqrt{2}$ which do not form a conjugate pair.
- ☑ If $b^2 - 4ac < 0$, then roots of equations are complex. If a, b and c are real then complex roots occur in conjugate pair like of the form $p + iq$ and $p - iq$. If all coefficients are not real then complex roots are not conjugate.

Illustration 11 If $a + b + c = 0$ then prove that the roots of the equation $4ax^2 + 3bx + 2c = 0$ where $a, b, c \in R$ are real and distinct

Solution

Given equation $4ax^2 + 3bx + 2c = 0$

$$D = (3b)^2 - 4(4a)(2c) = 9b^2 - 32ac = 9(-a - c)^2 - 32ac = 9a^2 - 14ac + 9c^2$$

$$= 9c^2\left(\left(\frac{a}{c}\right)^2 - \frac{14}{9}\frac{a}{c} + 1\right) = 9\left(\left(\frac{a}{c} - \frac{7}{9}\right)^2 - \frac{49}{81} + 1\right)$$

This is always positive.

Hence, roots are real and distinct.

Illustration 12 If $p, q \in \{1, 2, 3, 4\}$, then find the number of equations of the form $px^2 + qx + 1 = 0$ having real roots.

Solution

For real roots, we must have

Discriminant ≥ 0 $\Rightarrow q^2 - 4p \geq 0$ $\Rightarrow q^2 \geq 4p$

If $p = 1$, then $q^2 \geq 4p$ $\Rightarrow q^2 \geq 4$ $\Rightarrow q = 2, 3, 4$

$p = 2$, then $q^2 \geq 4p$ $\Rightarrow q^2 \geq 8$ $\Rightarrow q = 3, 4$

$p = 3$, then $q^2 \geq 4p \Rightarrow q^2 \geq 12 \Rightarrow q = 4$

$p = 4$, then $q^2 \geq 4p \Rightarrow q^2 \geq 16 \Rightarrow q = 4$

Thus, we see that there are 7 cases.

Illustration 13 For some non-zero, real and distinct a, b, c, the equation $(a^2 + b^2)x^2 - 2b(a + c)x + b^2 + c^2 = 0$ has non-zero real roots. Prove that one of these roots is also the root of the equation $a^2x^2 + a(c - b)x - bc = 0$.

Solution

$$(a^2 + b^2)x^2 - 2b(a + c)x + b^2 + c^2 = 0$$

$$D = 4b^2(a + c)^2 - 4(a^2 + b^2)(b^2 + c^2) = -4(b^4 - 2b^2ac + a^2c^2) = -4(b^2 - ac)^2$$

For real roots, $D \geq 0 \Rightarrow -4(b^2 - ac)^2 \geq 0 \Rightarrow b^2 - ac = 0$

So, roots are real and equal

Hence, each root equals to $\dfrac{2b(a+c) \pm 0}{2(a^2 + b^2)} = \dfrac{b(a+c)}{a^2 + ac} = \dfrac{b}{a}$.

This root satisfies the equation $a^2x^2 + a(c - b)x - bc = 0$.

Illustration 14 If $a, b, c \in R$ such that $a + b + c = 0$ and $a \neq c$ then prove that the roots of equation $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are real and distinct.

Solution

Given equation is

$$(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$$

or $(-2a)x^2 + (-2b)x + (-2c) = 0$

or $ax^2 + bx + c = 0$

$\Rightarrow$ $D = b^2 - 4ac = (-c - a)^2 - 4ac = (c - a)^2 > 0$

Hence, roots are real and distinct.

Alternatively, $a + b + c = 0$ means $x = 1$ satisfies the equation $ax^2 + bx + c = 0$.

Illustration 15 If $f(x) = ax^2 + bx + c$, $g(x) = -ax^2 + bx + c$ where $ac \neq 0$, then prove that $f(x) \cdot g(x) = 0$ has at least two real roots.

Solution

Let D_1 and D_2 be discriminates of $ax^2 + bx + c = 0$ and $-ax^2 + bx + c = 0$, respectively.

Then, $D_1 = b^2 - 4ac$, $D_2 = b^2 + 4ac$

Now, $ac \neq 0$

So, either $ac > 0$ or $ac < 0$

If $ac > 0$, then $D_2 > 0$.

Therefore, roots of $-ax^2 + bx + c = 0$ are real.

If $ac < 0$, then $D_1 > 0$.

Therefore, roots of $ax^2 + bx + c = 0$ are real.

Thus, $f(x) \cdot g(x) = 0$ has at least two real roots.

General Quadratic Expression in Two Variables

General quadratic expression in two variables is givn by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \quad \text{...(i)}$$

Corresponding equation is:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Or $ax^2 + 2(hy + g)x + by^2 + 2fy + c = 0$ ...(ii)

$\therefore \quad x = \dfrac{-2(hy+g) \pm \sqrt{4(hy+g)^2 - 4a(by^2+2fy+c)}}{2a}$

$\Rightarrow \quad x = \dfrac{-(hy+g) \pm \sqrt{h^2y^2+g^2+2ghy-aby^2-2afy-ac}}{a}$

$\Rightarrow \quad ax+hy+g = \pm\sqrt{(h^2-ab)y^2+2(gh-af)y+g^2-ac}$ ...(iii)

Now, expression (i) can be resolved into two linear factors if $(h^2-ab)y^2+2(gh-af)y+g^2-ac$ is a perfect square and $h^2-ab>0$.
But $(h^2-ab)y^2+2(gh-af)y+g^2-ac$ will be a perfect square if the discriminant of the corresponding equation

$(h^2-ab)y^2+2(gh-af)y+g^2-ac=0$ is zero and $h^2-ab>0$.

$\Rightarrow \quad 4(gh-af)^2-4(h^2-ab)(g^2-ac)=0$ and $h^2-ab>0$

$\Rightarrow \quad g^2h^2+a^2f^2-2afgh-h^2g^2+abg^2+ach^2-a^2bc=0$ and $h^2-ab>0$

$\Rightarrow \quad abc+2fgh-af^2-bg^2-ch^2=0$ and $h^2-ab>0$

This is the required condition.

Illustration 16 Find the values of m for which the expression $2x^2+mxy+3y^2-5y-2$ can be resolved into two rational linear factors.

Solution

Given expression is

$2x^2+mxy+3y^2-5y-2$ ...(i)

Let us compare the coefficients of the given expression (i) with those of general expression $ax^2+2hxy+by^2+2gx+2fy+c$.

We get $a=2,\ h=\dfrac{m}{2},\ b=3,\ g=0,\ f=\dfrac{-5}{2},\ c=-2$

Expression $2x^2+mxy+3y^2-5y-2$ will have two linear factors if and only if

$abc+2fgh-af^2-bg^2-ch^2=0$

or $\quad 2\cdot 3(-2)+2\left(\dfrac{-5}{2}\right)(0)\left(\dfrac{m}{2}\right)-2\left(-\dfrac{5}{2}\right)^2-3\cdot 0^2-(-2)\left(\dfrac{m}{2}\right)^2=0$

or $\quad -12-\dfrac{25}{2}+\dfrac{m^2}{2}=0$

or $\quad m^2=49$

$\therefore \quad m=\pm 7$

■ RELATION BETWEEN ROOTS AND COEFFICIENTS OF QUADRATIC EQUATION

Let α and β be roots of quadratic equation $ax^2+bx+c=0$ then by factor theorem, we get

$ax^2+bx+c=a(x-\alpha)(x-\beta)$

$\qquad = a(x^2-(\alpha+\beta)x+\alpha\beta)$

Comparing coefficients, we have $\alpha+\beta=-b/a$ and $\alpha\beta=c/a$

Thus, we find that

$$\alpha+\beta=\frac{-b}{a}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

and $$\alpha\beta=\frac{c}{a}=\frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Also, if sum of roots is S and product is P, then quadratic equation is given by $x^2-Sx+P=0$.

Illustration 17 If α and β are the roots of $x^2 - a(x-1) + b = 0$, then find the value of $\dfrac{1}{\alpha^2 - a\alpha} + \dfrac{1}{\beta^2 - a\beta} + \dfrac{2}{a+b}$.

Solution

It is given that α and β are the roots of $x^2 - a(x-1) + b = 0$.

Then $\alpha^2 - a\alpha + a + b = 0$ and $\beta^2 - a\beta + a + b = 0$

$\therefore$ $\alpha^2 - a\alpha = \beta^2 - a\beta = -a - b$

Now, $\dfrac{1}{\alpha^2 - a\alpha} + \dfrac{1}{\beta^2 - a\beta} + \dfrac{2}{a+b} = \dfrac{1}{-(a+b)} + \dfrac{1}{-(a+b)} + \dfrac{2}{a+b} = 0$

Illustration 18 If the ratio of the roots of $ax^2 + 2bx + c = 0$ is same as the ratio of the roots of the $px^2 + 2qx + r = 0$, then prove that $\dfrac{b^2}{ac} = \dfrac{q^2}{pr}$.

Solution

Let roots of the equation $ax^2 + 2bx + c = 0$ be α and β. Also, let the roots of the equation $px^2 + 2qx + r = 0$ are γ and δ.

Given $\dfrac{\alpha}{\beta} = \dfrac{\gamma}{\delta} \Rightarrow \dfrac{\alpha}{\gamma} = \dfrac{\beta}{\delta}$

$$\Rightarrow \quad \frac{\alpha+\beta}{\gamma+\delta} = \sqrt{\frac{\alpha\beta}{\gamma\delta}} \Rightarrow \frac{-\frac{2b}{a}}{-\frac{2q}{p}} = \sqrt{\frac{\frac{c}{a}}{\frac{r}{p}}} \Rightarrow \frac{b^2}{ac} = \frac{q^2}{pr}$$

Illustration 19 If pth, qth and rth terms of an A.P. are in G.P. whose common ratio is k, then find the roots of equation $(q-r)x^2 + (r-p)x + (p-q) = 0$.

Solution

Given, $k = \dfrac{a+(q-1)d}{a+(p-1)d} = \dfrac{a+(r-1)d}{a+(q-1)d}$, where a is first term of A.P. and d is common ratio.

$$\Rightarrow \quad k = \frac{a+(q-1)d - a - (r-1)d}{a+(p-1)d - a - (q-1)d} = \frac{(q-r)d}{(p-q)d} = \frac{q-r}{p-q}$$

Since one root is 1, therefore, other root is $\dfrac{p-q}{q-r} = \dfrac{1}{k}$.

Illustration 20 If the equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$, then find the condition for this.

Solution

Let α and β be the roots of the equation $ax^2 + bx + c = 0$.

$\therefore$ $\alpha + \beta = -\dfrac{b}{a}$; $\alpha\beta = \dfrac{c}{a}$

According to the question, roots of equation $2x^2 + 8x + 2 = 0$ will be $\alpha - 1$ and $\beta - 1$.

Sum of roots $= -\dfrac{8}{2}$

$\Rightarrow$ $\alpha + \beta - 2 = -4$

$\Rightarrow$ $-\dfrac{b}{a} - 2 = -4 \Rightarrow \dfrac{b}{a} = 2$

Product of roots = 1

$\Rightarrow \quad (\alpha - 1)(\beta - 1) = 1$

$\Rightarrow \quad \alpha\beta - (\alpha + \beta) + 1 = 1$

$\Rightarrow \quad \frac{c}{a} + \frac{b}{a} + 1 = 1$

$\Rightarrow \quad c + b = 0 \Rightarrow b = -c$

Hence, $b = 2a = -c$

Illustration 21 If one root of $x^2 - x - k = 0$ is square of the other, then find the value of k.

Solution

Let α and α^2 be the roots of $x^2 - x - k = 0$.

Then $\alpha + \alpha^2 = 1$ and $\alpha^3 = -k$.

$\Rightarrow \quad (-k)^{1/3} + (-k)^{2/3} = 1 \Rightarrow -k^{1/3} + k^{2/3} = 1$

$\Rightarrow \quad (k^{2/3} - k^{1/3})^3 = 1 \Rightarrow k^2 - k - 3k(k^{2/3} - k^{1/3}) = 1$

$\Rightarrow \quad k^2 - k - 3k(1) = 1 \Rightarrow k^2 - 4k - 1 = 0$

$\Rightarrow \quad k = 2 \pm \sqrt{5}.$

Illustration 22 If the roots of the equation $x^2 - ax + b = 0$ are real and differ by a quantity which is less than c ($c > 0$), then prove that b lies between $\frac{a^2 - c^2}{4}$ and $\frac{a^2}{4}$.

Solution

Given roots are real and distinct, then

$a^2 - 4b > 0 \Rightarrow b < a^2/4$ ...(i)

Also, given that α and β differ by a quantity less than c.

$\Rightarrow \quad |\alpha - \beta| < c$ or $(\alpha - \beta)^2 < c^2 \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta < c^2$

or $\quad a^2 - 4b < c^2$ or $\frac{a^2 - c^2}{4} < b$ ...(ii)

$\Rightarrow \quad \frac{a^2 - c^2}{4} < b < \frac{a^2}{4}$ [Using (i) and (ii)]

Illustration 23 If a and b are non-zero distinct roots of $x^2 + ax + b = 0$, then find the least value of $x^2 + ax + b$.

Solution

$$x^2 + ax + b = \left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4} \geq b - \frac{a^2}{4}$$

Least value of given expression is $b - \frac{a^2}{4}$ when $x + \frac{a}{2} = 0$.

Since a and b are roots of given equation, we have

$a + b = -a,\ ab = b$

$\Rightarrow \quad b = -2a$ and $b(1 - a) = 0$

$\Rightarrow \quad a = 1$ as $b \neq 0$ and hence, $b = -2$

Hence, least value is:

$$b - \frac{a^2}{4} = -2 - \frac{1}{4} = -\frac{9}{4}$$

Illustration 24 If $\tan A$ and $\tan B$ are the roots of the quadratic equation $x^2 - px + q = 0$, then find the value of $\sin^2(A + B)$.

Solution

We have, $\tan A + \tan B = p$ and $\tan A \tan B = q$

$$\therefore \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{p}{1-q}$$

Now, $$\sin^2(A+B) = \frac{\tan^2(A+B)}{\sec^2(A+B)} = \frac{p^2/(1-q)^2}{1+p^2/(1-q)^2} = \frac{p^2}{(1-q)^2+p^2}$$

Illustration 25 If α and β are the roots of the equation $ax^2 + bx + c = 0$, then find the roots of the equation $ax^2 - bx(x-1) + c(x-1)^2 = 0$ in terms of α and β.

Solution

$$ax^2 - bx(x-1) + c(x-1)^2 = 0$$

$$\Rightarrow \quad \frac{ax^2}{(1-x)^2} + \frac{bx}{1-x} + c = 0 \qquad \text{...(i)}$$

α is a root of $ax^2 + bx + c = 0$.

Then let $\alpha = \frac{x}{1-x} \Rightarrow x = \frac{\alpha}{\alpha+1}$

So, roots of (i) are $\frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta}$.

Illustration 26 If α and β; α and γ; α and δ are the roots of the equations $ax^2 + 2bx + c = 0$, $2bx^2 + cx + a = 0$ and $cx^2 + ax + 2b = 0$, respectively, where a, b and c are positive real numbers, then prove that $\alpha + \alpha^2 = 0$.

Solution

Here $\quad \alpha + \beta = -\frac{2b}{a}, \gamma + \alpha = -\frac{c}{2b}, \alpha + \delta = -\frac{a}{c}$

and $\quad \alpha\beta = \frac{c}{a}, \alpha\gamma = \frac{a}{2b}, \alpha\delta = \frac{2b}{c}$

$$\Rightarrow \quad \alpha + \delta = -\frac{1}{\alpha\beta}$$

or $$\alpha^2\beta + \alpha\beta\delta = -1 \qquad \text{...(i)}$$

$$\alpha + \beta = -\frac{1}{\alpha\gamma}$$

or $$\alpha^2\gamma + \alpha\beta\gamma = -1 \qquad \text{...(ii)}$$

$$\alpha + \gamma = -\frac{1}{\alpha\delta},$$

$$\alpha^2\delta + \alpha\delta\gamma = -1 \qquad \text{...(iii)}$$

Solving equations (i), (ii) and (iii), we get $\alpha = -1$

$$\therefore \quad \alpha + \alpha^2 = (-1) + (-1)^2 = -1 + 1 = 0$$

Illustration 27 Let α and γ be the roots of the equation $Ax^2 - 4x + 1 = 0$. Also, let β and δ the roots of the equation of $Bx^2 - 6x + 1 = 0$. Then find the values of A and B such that α, β, γ and δ are in H.P.

Solution

$\alpha, \beta, \gamma, \delta$ are in H.P.

$\Rightarrow \quad \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ are in A.P.

And these may be taken as $a - 3d, a - d, a + d, a + 3d$.

Replacing x by $1/x$ we get the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$

So, $\quad x^2 - 4x + A = 0$ has roots $a - 3d, a + d$.

And $\quad x^2 - 6x + B = 0$ has roots $a - d, a + 3d$

From sum of roots $2(a - d) = 4$ and $2(a + d) = 6$

$\therefore \quad a = 5/2, d = 1/2$

From product of roots, we get $(a - 3d)(a + d) = A = 3$

And $\quad (a - d)(a + 3d) = B = 8$

Concept Application Exercise 2

1. For which of the following values of a, the roots of the equation $x^2 + a^2 = 8x + 6a$ are real?
   (1) $[-8, 2]$ (2) $[-2, 8]$ (3) $(-3, 8)$ (4) $(2, 9)$
2. Find the condition if the roots of $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{ac}\,x + b = 0$ are simultaneously real.
3. If $a < c < b$, then check the nature of roots of the equation $(a - b)^2 x^2 + 2(a + b - 2c)x + 1 = 0$.
4. If $b_1 b_2 = 2(c_1 + c_2)$, then prove that at least one of the equations $x^2 + b_1 x + c_1 = 0$ and $x^2 + b_2 x + c_2 = 0$ has real roots.
5. The equation $x^2 + bx + c = 0$ has distinct roots. If 2 is subtracted from each root, then the resultant roots are reciprocals of the original roots. The value of $b^2 + c^2$ is
   (1) -5 (2) 3 (3) 4 (4) 5
6. If $ab + bc + ca = 0$, then solve the equation $a(b - 2c)x^2 + b(c - 2a)x + c(a - 2b) = 0$.
7. If the sum of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is -1, then find the product of the roots.
8. If x_1 and x_2 are the roots of $x^2 + (\sin\theta - 1)x - \frac{1}{2}\cos^2\theta = 0$, then find the maximum value of $x_1^2 + x_2^2$.
9. The sum of the roots of an equation is 2 and sum of their cubes is 98. Find the equation.
10. If α and β are the roots of the equation $x^2 - ax + b = 0$ and $A_n = \alpha^n + \beta^n$, then prove that $A_{n+1} = aA_n - bA_{n-1}$.
11. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio 2 : 3, then find the value of m.
12. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the nth power of the other, then prove that $(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} = -b$.
13. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then prove that $P(x)\,Q(x) = 0$ has at least two real roots.
14. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$. Also, let the roots of the equation $x^2 - 10ax - 11b = 0$ be c and d. Then the value of $a + b + c + d$, where $a \neq b \neq c \neq d$ is
    (1) 121 (2) 1210 (3) 1000 (4) 1200
15. Let a, b and c be real numbers with $a \neq 0$. Also, let α and β be the roots of the equations $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α and β.

COMMON ROOTS

Condition for One Common Root

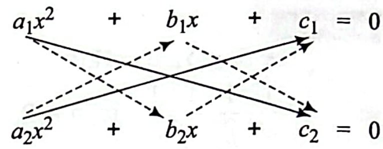
Let α be the common root of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$.

Then, $a_1\alpha^2 + b_1\alpha + c_1 = 0$ and $a_2\alpha^2 + b_2\alpha + c_2 = 0$

Solving these two equations by cross-multiplication, we get

$$\frac{\alpha^2}{b_1 c_2 - b_2 c_1} = \frac{\alpha}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1}$$

$$\Rightarrow \quad \alpha^2 = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} \text{ and } \alpha = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}$$



$$\Rightarrow \quad \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} = \left(\frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right)^2$$

$$\Rightarrow \quad (c_1 a_2 - c_2 a_1)^2 = (b_1 c_2 - b_2 c_1)(a_1 b_2 - a_2 b_1)$$

This is the required condition for one root to be common of two quadratic equations.
There is a trick to memorize this condition, which is shown in the following figure:
(Cross product of extremes)^2 = Product of cross products of middle terms with extremes

The common root is given by $\alpha = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}$ *or* $\alpha = \frac{b_1 c_2 - b_2 c_1}{c_1 a_2 - c_2 a_1}$.

Note:

- ☑ The common root can also be obtained by making the coefficient of x^2 common to two given equations and then subtracting the two equations. The other roots of the given equations can be determined by using the relations between their roots and coefficients.

Condition for Both Roots being Common

Let α and β be the common roots of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$.

Then both the equations are identical. Hence, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Note:

- ☑ If two quadratic equations with real coefficients have a non-real complex common root then both roots will be common, i.e., both the equations will be the same. So the coefficients of the corresponding powers of x will have proportional values.
- ☑ If two quadratic equations with rational coefficients have a common irrational root $p + \sqrt{q}$ then both roots will be common, i.e., no two different quadratic equations with rational coefficients can have a common irrational root $p + \sqrt{q}$.

Illustration 28 Find the values of m for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

Solution

Let α be the common root of the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$.
Then, α must satisfy both the equations

$\therefore \quad 3\alpha^2 + 4m\alpha + 2 = 0$ ...(i)

and $\quad 2\alpha^2 + 3\alpha - 2 = 0$ ...(ii)

From equation (ii), $(2\alpha - 1)(\alpha + 2) = 0$

$\therefore \quad \alpha = 1/2$ or -2

So, common root may be 1/2 or -2.

Putting $\alpha = 1/2$ and -2 in equation (i), alternatively, we get m as $-\frac{11}{8}$ and $\frac{7}{4}$, respectively.

Illustration 29 If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have a common root and $a, b, c \in N$ then find minimum value of $a + b + c$.

Solution

Roots of $x^2 + 3x + 5 = 0$ are non-real. Thus, given equations will have two common roots.

$$\Rightarrow \quad \frac{a}{1} = \frac{b}{3} = \frac{c}{5} = \lambda$$

$$\Rightarrow \quad a + b + c = 9\lambda$$

Thus, minimum value of $a + b + c = 9$

Illustration 30 Given that α_1, β_1 are the roots of $ax^2 + bx + c = 0$ and α_2, β_2 are the roots of $px^2 + qx + r = 0$. If $\alpha_1\alpha_2 = \beta_1\beta_2 = 1$, then prove that $\frac{a}{r} = \frac{b}{q} = \frac{c}{p}$.

Solution

$$\alpha_2 = \frac{1}{\alpha_1} \text{ and } \beta_2 = \frac{1}{\beta_1}$$

In other words, the roots of the second equation are reciprocals of the roots of the first one.

In equation (i), replacing x by $\frac{1}{x}$, we get $cx^2 + bx + a = 0$ whose roots are α_2, β_2.

Hence, it is same as $px^2 + qx + r = 0$.

Comparing ratios of coefficients, we get $\frac{c}{p} = \frac{b}{q} = \frac{a}{r}$

Illustration 31 If $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$, $(a \neq b)$ have a common root, then find the equation formed by their other roots.

Solution

Subtracting the given equations, we get

$$(a - b)x + c(b - a) = 0$$

So, $\quad x = c$ is the common root.

Thus, roots of $x^2 + ax + bc = 0$ are b and c and those of $x^2 + bx + ca = 0$ are c and a.

Also, $\quad b + c = -a$.

Thus, the required equation is:

$$x^2 - (a + b)x + ab = 0$$

$$\Rightarrow \quad x^2 + cx + ab = 0$$

Repeated Roots

If equation $f(x) = 0$, where $f(x)$ is a polynomial function, and if it has roots $\alpha, \alpha, \beta, \ldots$ or root α is repeated root, then $f(x) = 0$ is equivalent to $(x - \alpha)^2 (x - \beta) \ldots = 0$, from which we can conclude that

$$f'(x) = 0 \text{ or } 2(x - \alpha)[(x - \beta)\ldots] + (x - \alpha)^2 [(x - \beta)\ldots]' = 0$$

or $\quad (x - \alpha)[2\{(x - \beta)\ldots\} + (x - \alpha)\{(x - \beta)\ldots\}'] = 0$ has root α.

Thus, if α occurs twice in the equation $f(x) = 0$ then it is common in the equations $f(x) = 0$ and $f'(x) = 0$.

Similarly, if α occurs thrice in the equation then it is common in the equations $f(x) = 0$, $f'(x) = 0$ and $f''(x) = 0$.

RELATION BETWEEN COEFFICIENTS AND ROOTS OF EQUATION OF DEGREE *n*

- Let α and β be that roots of quadratic equation $ax^2 + bx + c = 0$. Then by factor theorem,
  $ax^2 + bx + c = a(x - \alpha)(x - \beta) = a(x^2 - (\alpha + \beta)x + \alpha\beta)$
  Comparing coefficients, we have $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$

- Let α, β and γ be the roots of cubic equation $ax^3 + bx^2 + cx + d = 0$. Then
  $ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma) = a(x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma)$
  Comparing coefficients, we get
  $\alpha + \beta + \gamma = -b/a$, $\alpha\beta + \beta\gamma + \alpha\gamma = c/a$, and $\alpha\beta\gamma = -d/a$
- If α, β, γ and δ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then
  $\alpha + \beta + \gamma + \delta = -b/a$
  $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c/a$ (sum of products taking two at a time)
  $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -d/a$ (sum of products taking three at a time)
  $\alpha\beta\gamma\delta = e/a$

In general, if $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$, then

Sum of all roots, $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n = -\dfrac{a_1}{a_0}$

Sum of product of roots taking two at a time $= \dfrac{a_2}{a_0}$

Sum of product of roots taking three at a time $= -\dfrac{a_3}{a_0}$

… …

… …

Product of all roots $= (-1)^n \dfrac{a_n}{a_0}$

Note:

- ☑ A polynomial equation of degree n has n roots (real or imaginary).
- ☑ If all the coefficients are real then the imaginary roots occur in pairs i.e., number of imaginary roots is always even.
- ☑ If the degree of a polynomial equation is odd then the number of real roots will also be odd. It follows that at least one of the roots will be real.
- ☑ **Solving Cubic Equation:**

  By using factor theorem, together with some intelligent guessing, we can factorize polynomial of higher degree. To solve a cubic equation $ax^3 + bx^2 + cx + d = 0$, obtain one factor $(x - \alpha)$ by trial and error.
  Factorize $ax^3 + bx^2 + cx + d$ as $(x - \alpha)(hx^2 + kx + s)$, then solve the quadratic equation $(hx^2 + kx + s = 0)$ for other roots.

Illustration 32 If two roots of $x^3 - ax^2 + bx - c = 0$ are equal in magnitude but opposite in signs then find the condition.

Solution

Let the roots be x_1, $-x_1$ and x_2.

Then $x_1 - x_1 + x_2 = a \Rightarrow x_2 = a$

Hence, $x = a$ is a root of the given equation.

$\Rightarrow$ $a^3 - a^3 + ab - c = 0 \Rightarrow ab = c$

Illustration 33 If α, β and γ are the roots of the equation $x^3 + 4x + 1 = 0$, then find the value of $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$

Solution

For given equation

$\alpha + \beta + \gamma = 0$,

$\alpha\beta + \beta\gamma + \alpha\gamma = 4$ and $\alpha\beta\gamma = -1$

Now, $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1} = (-\gamma)^{-1} + (-\beta)^{-1} + (-\alpha)^{-1}$

$$= -\frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma} = 4$$

Illustration 34 If α, β, γ and δ are the roots of the equation $x^4 - Kx^3 + Kx^2 + Lx + M = 0$, where K, L and M are real numbers, then find the minimum value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$.

Solution

Roots of equation $x^4 - Kx^3 + Kx^2 + Lx + M = 0$ are α, β, γ and δ.

$$\sum \alpha = K, \sum \alpha\beta = K, \sum \alpha\beta\gamma = -L, \alpha\beta\gamma\delta = M$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2$$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2\sum \alpha\beta$$

$$= K^2 - 2K = (K-1)^2 - 1$$

$\therefore$ $(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)_{min.} = -1$

Concept Application Exercise 3

1. If a, b, c, a_1, b_1 and c_1 are rational and equations $ax^2 + 2bx + c = 0$ and $a_1x^2 + 2b_1x + c_1 = 0$ have one and only one root in common, then prove that both $b^2 - ac$ and $b_1{}^2 - a_1c_1$ are perfect squares.
2. Every pair from among the equations $x^2 + ax + bc = 0$, $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ has a common root. Then find the sum and product of common roots.
3. If a, b, p and q are non-zero real numbers, then how many common roots can the equations $2a^2x^2 - 2\,ab\,x + b^2 = 0$ and $p^2x^2 + 2\,pq\,x + q^2 = 0$ have?
4. If the equations $ax^2 + bx + c = 0$ and $x^3 + 3x^2 + 3x + 2 = 0$ have two common roots, then find the condition.
   (1) $a = b = c$ (2) $a + b = c$ (3) $a = -b = c$ (4) $a = 2b = 3c$
5. If $\tan\theta_1$, $\tan\theta_2$ and $\tan\theta_3$ are the real roots of the $x^3 - (a+1)x^2 + (b-a)x - b = 0$, where $\theta_1 + \theta_2 + \theta_3 \in (0, \pi)$, then find the value of $\theta_1 + \theta_2 + \theta_3$.
6. Let r, s and t be the roots of the equation, $8x^3 + 1001x + 2008 = 0$. Then find the value of $(r+s)^3 + (s+t)^3 + (t+r)^3$.
7. If α, β and γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, then find the value of $\left(\alpha - \frac{1}{\beta\gamma}\right)\left(\beta - \frac{1}{\gamma\alpha}\right)\left(\gamma - \frac{1}{\alpha\beta}\right)$.

QUADRATIC FUNCTION

Let $f(x) = ax^2 + bx + c$, where $a, b, c, \in R$ and $a \neq 0$.

We have

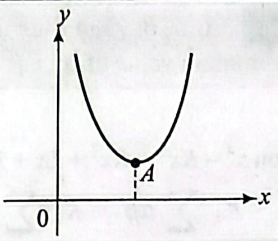
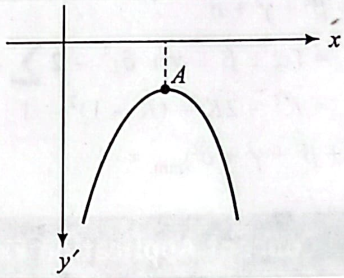
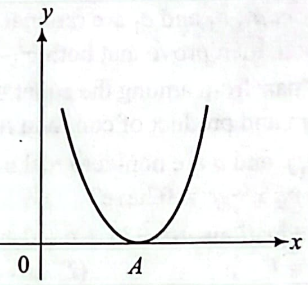
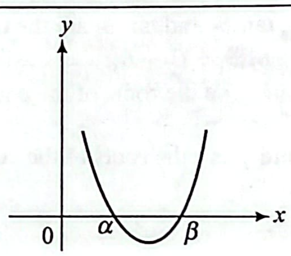
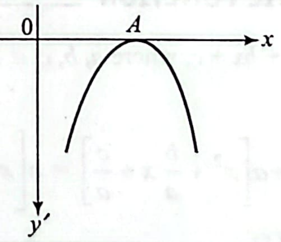
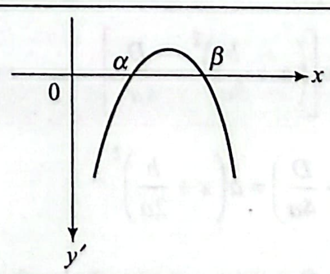
$$f(x) = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

Now, let $y = f(x)$

$$\Rightarrow \quad y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

$$\Rightarrow \quad \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

Thus, $y = f(x)$ represent a parabola whose axis is parallel to y-axis and vertex is at $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$. For some values of x, $f(x)$ may be positive, negative or zero. Also, for $a > 0$, the parabola cpens upwards and for $a < 0$, the parabola opens downwards. This gives the following cases:

(i)	$a > 0$ and $D < 0$ So, $f(x) > 0\ \forall\ x \in R$ i.e., $f(x)$ is positive for all values of x. Range of function is $\left[-\dfrac{D}{4a}, \infty\right)$, and $x = -\dfrac{b}{2a}$ is point of minima.	
(ii)	$a < 0$ and $D < 0$ So, $f(x) < 0\ \forall\ x \in R$ i.e., $f(x)$ is negative for all values of x. Range of function is $\left(-\infty, -\dfrac{D}{4a}\right]$, and $x = -\dfrac{b}{2a}$ is point of maxima.	
(iii)	$a > 0$ and $D = 0$ So, $f(x) \geq 0\ \forall\ x \in R$ i.e., $f(x)$ is positive for all values of x except at vertex where $f(x) = 0$.	
(iv)	$a > 0$ and $D > 0$ Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$. Then $f(x) > 0\ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0\ \forall\ x \in (\alpha, \beta)$.	
(v)	$a < 0$ and $D = 0$ So, $f(x) \leq 0\ \forall\ x \in R$ i.e., $f(x)$ is negative for all values of x except at vertex where $f(x) = 0$.	
(vi)	$a < 0$ and $D > 0$ Let $f(x) = 0$ have two roots α and β $(\alpha < \beta)$. Then $f(x) < 0\ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0\ \forall\ x \in (\alpha, \beta)$	

Note:

☑ If $f(x) \geq 0$, $\forall\ x \in R$ then $a > 0$ and $D \leq 0$

☑ If $f(x) \leq 0\ \forall\ x \in R$ then $a < 0$ and $D \leq 0$

Illustration 35 Let $f(x) = ax^2 + bx + c$ be a quadratic expression having its vertex at (3, –2) and $f(0) = 10$. Then find $f(x)$.

Solution

$$f(x) = ax^2 + bx + c$$
$$f(0) = 10$$
$$\Rightarrow \quad c = 10$$

Vertex $\equiv \left(-\dfrac{b}{2a}, -\dfrac{D}{4a}\right)$

$$\Rightarrow \quad -\frac{b}{2a} = 3 \qquad \text{...(i)}$$

and $$-\frac{(b^2 - 4ac)}{4a} = -2 \qquad \text{...(ii)}$$

From (ii), we get

$$-\frac{b^2}{4a} + 10 = -2 \qquad \text{(Putting } c = f(0) = 10\text{)}$$

$$\Rightarrow \quad -\frac{b}{2a} \cdot \frac{b}{2} = -12 \Rightarrow \frac{3b}{2} = -12 \Rightarrow b = -8$$

$$\therefore \quad a = \frac{4}{3}$$

$$\therefore \quad f(x) = \frac{4}{3}x^2 - 8x + 10$$

Illustration 36 If the equation $ax^2 + bx + c = x$ has no real roots then prove that the equation $a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c = x$ will have no real root.

Solution

It is given that $ax^2 + bx + c = x$ has no real roots.

$\Rightarrow \quad ax^2 + bx + c > x$ or $ax^2 + bx + c < x$ for all real x

Let $\quad ax^2 + bx + c > x$ and $ax^2 + bx + c = P(x)$

$\therefore \quad P(x) > x$

$\Rightarrow \quad P(P(x)) > P(x)$

$\Rightarrow \quad a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c > ax^2 + bx + c > x$

Similarly, for $ax^2 + bx + c < x$, it can be proved that

$$a(ax^2 + bx + c) + b(ax^2 + bx + c) + c < 0$$

Hence, given equation has no real solution.

Illustration 37 For all $x \in R$, if $\lambda x^2 - 9\lambda x + 5\lambda + 1 > 0$, then find the values of λ.

Solution

$$\lambda x^2 - 9\lambda x + 5\lambda + 1 > 0 \text{ for all } x \in R$$

Then we must have $\lambda \geq 0$ and $D < 0$.

$\Rightarrow \quad \lambda \geq 0$ and $(9\lambda)^2 - 4\lambda(5\lambda + 1) < 0$

$\Rightarrow \quad \lambda \geq 0$ and $61\lambda^2 - 4\lambda < 0$

$$\Rightarrow \quad 0 \leq \lambda < \frac{4}{61}$$

Illustration 38 If $a, b \in R, a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then prove that $a + b + 1 > 0$.

Solution

Let $f(x) = ax^2 - bx + 1$. Since $f(x) = 0$ has imaginary roots,

$$D = b^2 - 4a < 0 \Rightarrow a > 0$$

So, the graph of given quadratic equation is concave upwards.

$\Rightarrow \quad f(x) > 0, \forall x \in R$

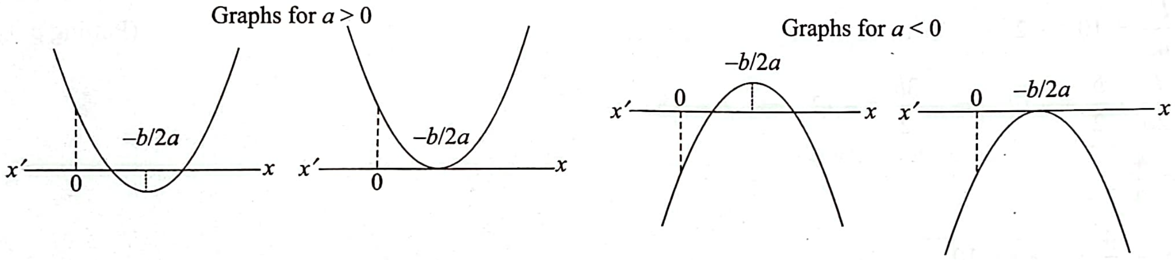
$\Rightarrow \quad f(-1) > 0$

$\Rightarrow \quad a + b + 1 > 0$

LOCATION OF ROOTS

In some problems, we want the roots α and β of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c. Consider the following cases.

1. $\alpha, \beta > 0$



Conditions:

(i) Sum of roots, $\alpha + \beta > 0$

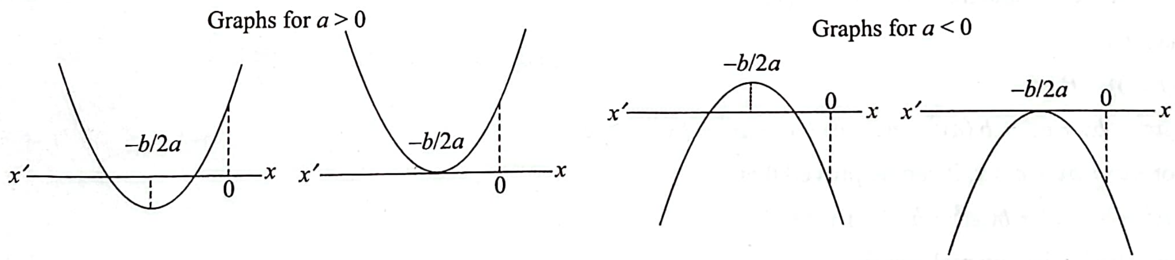
$\Rightarrow \quad -\dfrac{b}{2a} > 0$

(ii) Product of roots, $\alpha \cdot \beta > 0$

$\Rightarrow \quad a \cdot f(0) > 0$ $\quad (\because$ when $a > 0, f(0) > 0$ and when $a < 0, f(0) < 0)$

(iii) $D \geq 0$

2. $\alpha, \beta < 0$



Conditions:

(i) Sum of roots, $\alpha + \beta < 0$

$\Rightarrow \quad -\dfrac{b}{2a} < 0$

(ii) Product of roots, $\alpha \cdot \beta > 0$

$\Rightarrow \quad a \cdot f(0) > 0$

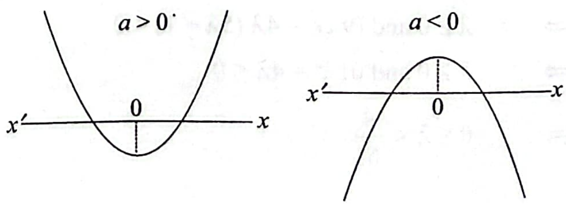
(iii) $D \geq 0$

3. $\alpha < 0 < \beta$ (roots of opposite signs)

Product of roots, $\alpha\beta < 0$

Alternatively

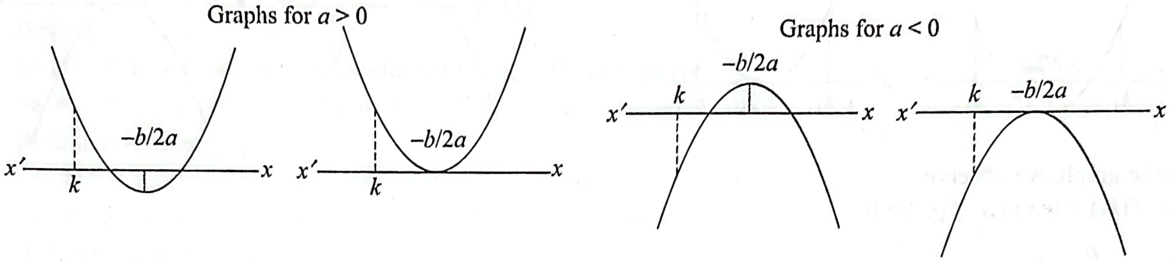
When $a > 0, f(0) < 0$



And when $a < 0, f(0) > 0$

$\Rightarrow \quad a \cdot f(0) < 0$

4. $\alpha, \beta > k$



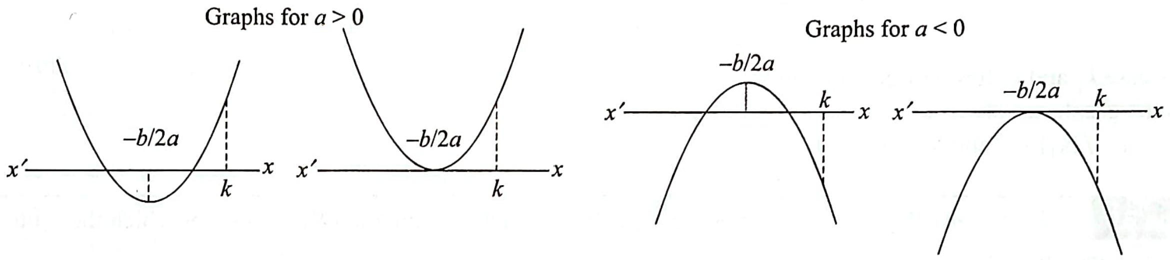
Conditions:

(i) $a \cdot f(k) > 0$ ($\because$ when $a > 0, f(k) > 0$ and when $a < 0, f(k) < 0$)

(ii) $-\dfrac{b}{2a} > k$

(iii) $D \geq 0$

5. $\alpha, \beta < k$



Conditions:

(i) $a \cdot f(k) > 0$ ($\because$ when $a > 0, f(k) > 0$ and when $a < 0, f(k) < 0$)

(ii) $-\dfrac{b}{2a} < k$

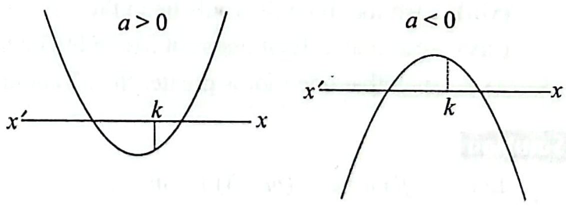
(iii) $D \geq 0$

6. $\alpha < k < \beta$ (one root is less than k and other root greater than k)

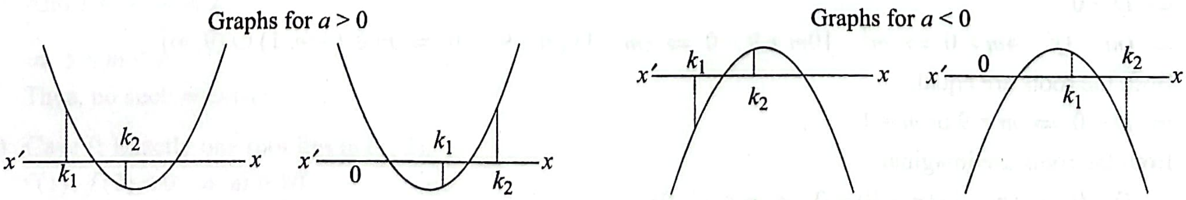
When $a > 0, f(k) < 0$

And when $a < 0, f(k) > 0$

$\Rightarrow \quad a \cdot f(k) < 0$



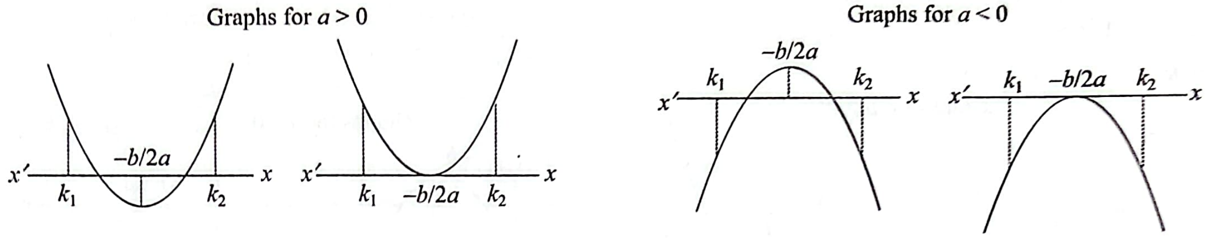
7. Exactly one root lies between (k_1, k_2)



From the graphs we can see that $f(k_1)$ and $f(k_2)$ have opposite signs.

Hence, $f(k_1) \cdot f(k_2) < 0$

8. Both the roots lie in the interval (k_1, k_2)



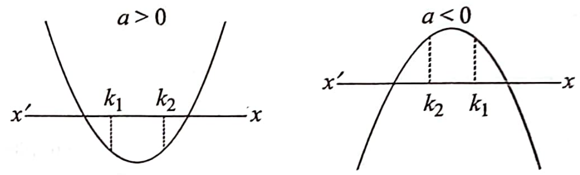
From the graph, we observe

(i) $a \cdot f(k_1) > 0$ and $a \cdot f(k_2) > 0$

(ii) $k_1 < -\dfrac{b}{2a} < k_2$

(iii) $D \geq 0$

9. One root is less than k_1 and other is greater than k_2



In this case k_1 and k_2 lies between the roots.
From the graph, we observe

$a \cdot f(k_1) < 0$ and $a \cdot f(k_2) < 0$

Illustration 39 Let $x^2 - (m-3)x + m = 0$ ($m \in R$) be a quadratic equation. Find the values of m for which the roots are

(i) real and distinct
(ii) equal
(iii) not real
(iv) opposite in sign
(v) equal in magnitude but opposite in sign
(vi) positive
(vii) negative
(viii) such that at least one root is positive
(ix) such that one root is smaller than 2 and the other root is greater than 2
(x) both greater than 2
(xi) both smaller than 2
(xii) such that exactly one roots lies in the interval (1, 2)
(xiii) such that both the roots lie in the interval (1, 2)
(xiv) such that at least one root lies in the interval (1, 2)
(xv) such that one root is greater than 2 and the other root is smaller than 1

Solution

Let $f(x) = x^2 - (m-3)x + m = 0$

(i) Both the roots are real and distinct.

$\Rightarrow D > 0$

$\Rightarrow (m-3)^2 - 4m > 0 \Rightarrow m^2 - 10m + 9 > 0 \Rightarrow (m-1)(m-9) > 0 \Rightarrow m \in (-\infty, 1) \cup (9, \infty)$

(ii) Both the roots are equal.

$\Rightarrow D = 0 \Rightarrow m = 9$ or $m = 1$

(iii) Both the roots are imaginary

$\Rightarrow D < 0 \Rightarrow (m-1)(m-9) < 0 \Rightarrow m \in (1, 9)$

(iv) The roots are opposite in sign.

i.e., product of roots is negative.

$\Rightarrow m < 0 \Rightarrow m \in (-\infty, 0)$

(v) Roots are equal in magnitude but opposite in sign
i.e., sum of roots is zero as well as $D \geq 0$.
$\Rightarrow m \in (-\infty, 1) \cup (9, \infty)$ and $m - 3 = 0$ i.e., $m = 3$
No such m exists. So, $m \in \phi$

(vi) Both the roots are positive.
i.e., $D \geq 0$ as well as sum and product of roots, both are positive.
$\Rightarrow m - 3 > 0, m > 0$ and $m \in (-\infty, 1] \cup [9, \infty) \Rightarrow m \in [9, \infty)$

(vii) Both the roots are negative.
i.e., $D \geq 0$ and sum is negative but product is positive.
$\Rightarrow m - 3 < 0, m > 0$ and $m \in (-\infty, 1] \cup [9, \infty) \Rightarrow m \in (0, 1]$

(viii) At least one root is positive.
This means that either one root is positive or both the roots are positive.
i.e., union of (iv) and (vi)
$\Rightarrow m \in (-\infty, 0) \cup [9, \infty)$

(ix) One root is smaller than 2 and other root is greater than 2.
i.e., 2 lies between the roots.
$\Rightarrow f(2) < 0$
$\Rightarrow 4 - 2(m - 3) + m < 0$
$\Rightarrow m > 10$

(x) Both the roots are greater than 2.
$\Rightarrow f(2) > 0, D \geq 0, -\dfrac{b}{2a} > 2$
$\Rightarrow m < 10$ and $m \in (-\infty, 1] \cup [9, \infty)$ and $m - 3 > 4$
$\Rightarrow m \in [9, 10)$

(xi) Both the roots are smaller than 2.
$\Rightarrow f(2) > 0, D > 0, -\dfrac{b}{2a} < 2$
$\Rightarrow m \in (-\infty, 1]$.

(xii) Exactly one root lies between (1, 2).
$\Rightarrow f(1) \cdot f(2) < 0$
$\Rightarrow 4(10 - m) < 0$
$\Rightarrow m \in (10, \infty)$

(xiii) Both the roots lie in the interval (1, 2).
Then
$D \geq 0$
$\Rightarrow m \leq 1$ or $m \geq 9$ ...(1)
Also, $f(1) > 0$ and $f(2) > 0$
$\Rightarrow 10 > m$ ...(2)
And $1 < -\dfrac{b}{2a} < 2$
$\Rightarrow 5 < m < 7$ ...(3)
Thus, no such m exists

(xiv) **Case I:** Exactly one root lies in (1, 2).
$f(1) \cdot f(2) < 0 \Rightarrow m > 10$
Case II: Both roots lie in (1, 2).
From (xiii), $m \in \phi$
For at least one root lying in (1, 2), we have $m \in (10, \infty)$.

(xv) One root is greater than 2 and other is smaller than 1.

$f(1) < 0$ ... (1)

$f(2) < 0$ ...(2)

From (1), $f(1) < 0$, but $f(1) = 4$, which is not possible

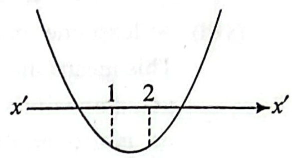
Thus, no such m exists.

Illustration 40 If $x^2 + 2ax + a < 0 \ \forall \ x \in [1, 2]$ then find the values of a.

Solution

$$x^2 + 2ax + a < 0, \forall \ x \in [1, 2]$$

So, 1 and 2 lie between the roots of the equation



$$x^2 + 2ax + a = 0$$

$\Rightarrow \quad f(1) < 0$ and $f(2) < 0$

$\Rightarrow \quad 1 + 2a + a < 0$ and $4 + 4a + a < 0$

$\Rightarrow \quad a < -\dfrac{1}{3}$ and $a < -\dfrac{4}{5}$

$\Rightarrow \quad a \in \left(-\infty, -\dfrac{4}{5}\right)$

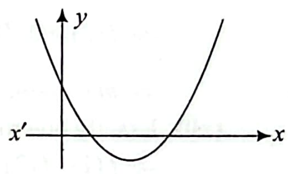
Illustration 41 Find all the possible values of a for which $ax^2 + (a - 3)\,x + 1 < 0$ for at least one positive real x.

Solution

Let $\quad f(x) = ax^2 + (a - 3)x + 1$

Case I: $a > 0$

If $a > 0$, then $f(x)$ will be negative only for those values of x which lie between the roots. We can say that $f(x)$ will be less than zero for at least one positive real x, when $f(x) = 0$ has distinct roots and at least one of these roots is positive real root



Since $f(0) = 1 > 0$, the favourable graph according to the questions will be as shown in the figure.

From the graph, we can see that both the roots are non-negative.

For this,

(i) $D > 0$

$\Rightarrow \quad (a - 3)^2 - 4a > 0$

$\Rightarrow \quad a < 1$ or $a > 9$ ...(i)

(ii) sum > 0 and product ≥ 0

$\Rightarrow \quad -(a - 3) > 0$ and $1/a > 0$

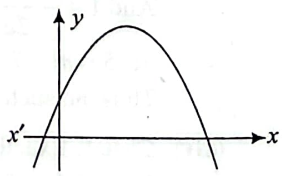
$\Rightarrow \quad a < 3$ ...(ii)

From (i) and (ii), $a < 1$

But $a > 0 \Rightarrow a \in (0, 1)$

Case II: $a < 0$

Since $f(0) = 1 > 0$, graph will be as shown in the figure.



From the graph, we get

$ax^2 + a\,(a - 3)\,x + 1 < 0$, for at least one positive x.

Case III: $a = 0$

If $a = 0$, $f(x) = -3x + 1$

$\Rightarrow f(x) < 0, \ \forall \ x > 1/3$

Thus, from all the cases the required set of values of a is $(-\infty, 1)$.

Concept Application Exercise 4

1. Find the values of λ for which $\dfrac{\lambda x^2+3x+4}{x^2+2x+2}<5$ for all $x \in R$.
2. Let $f(x) = ax^2 - bx + c^2$, $b \neq 0$ and $f(x) \neq 0$ for all $x \in R$. Then prove that $4a + c^2 > 2b$.
3. If α and β are the roots of $x^2 - 3x + \lambda = 0$ ($\lambda \in R$) and $\alpha < 1 < \beta$, then find the set of values of λ.
4. If the roots of the equation $(a-1)(x^2+x+1)^2 = (a+1)(x^4+x^2+1)$ are real and distinct then find the values of a.
5. x_1 and x_2 are the roots of $ax^2 + bx + c = 0$ and $x_1 \cdot x_2 < 0$. Then prove that roots of $x_1(x-x_2)^2 + x_2(x-x_1)^2 = 0$ are real and of opposite signs.
6. If $a \in R$, $b \in R$ then prove that the equation $x^2 - abx - a^2 = 0$ has one positive root and one negative root.
7. Find the all the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4.
8. If the roots of the quadratic equation, $(4p - p^2 - 5)x^2 - (2p-1)x + 3p = 0$ lie on either side of unity then the integral values of p are
   (1) $\{1, 4\}$ (2) $\{2, 3\}$ (3) $\{1, 3\}$ (4) $\{2, 4\}$
9. For the quadratic equation $4x^2 - 2(a+c-1)x + ac - b = 0$ $(a > b > c)$, prove that exactly one of the roots lies between $\dfrac{c}{2}$ and $\dfrac{a}{2}$.
10. If $a < b < c < d$, then prove that the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are real and distinct.
11. Let a, b and c be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that $1 + \dfrac{c}{a} + \left|\dfrac{b}{a}\right| < 0$
12. If $x^2 + (a-b)x + (1-a-b) = 0$ where $a, b \in R$ then find the values of a for which equation has unequal real roots for all values of b.

SOLVED EXAMPLES

MISCELLANEOUS PROBLEMS BASED ON ONE OR MORE THAN ONE CONCEPTS

Example 1 If the roots of an equation $x^n - 1 = 0$ are 1, $a_1, a_2, \ldots, a_{n-1}$, then the value of $(1 - a_1)(1 - a_2)(1 - a_3) \ldots (1 - a_{n-1})$ is

(1) n (2) n^2
(3) n^n (4) 0

Solution

Clearly,

$$x^n - 1 = (x - 1)(x - a_1)(x - a_2) \ldots (x - a_{n-1})$$

$\Rightarrow$ $$\frac{x^n - 1}{x - 1} = (x - a_1)(x - a_2) \ldots (x - a_n)$$

$\Rightarrow$ $1 + x + x^2 + \ldots + x^n = (x - a_1)(x - a_2) \ldots (x - a_{n-1})$

Putting $x = 1$, we get

$\Rightarrow$ $n = (1 - a_1)(1 - a_2) \ldots (1 - a_{n-1})$

Hence, the correct answer is (1).

Example 2 The number of real solutions of the equation $(9/10)^x = -3 + x - x^2$ is

(1) 2 (2) 0
(3) 1 (4) none of these

Solution

Let $f(x) = -3 + x - x^2$. Then $f(x) < 0$ for all x, because coefficient of $x^2 < 0$ and $D < 0$.

Thus, L.H.S. of the given equation is always positive whereas the R.H.S. is always less than zero. Hence, there is no solution.

Hence, the correct answer is (2).

Example 3 The integral values of m for which the roots of the equation $mx^2 + (2m - 1)x + (m - 2) = 0$ are rational can be given by the expression ($n \in I$)

(1) n^2 (2) $n(n + 2)$
(3) $n(n + 1)$ (4) none of these

Solution

Discriminant, $D = (2m - 1)^2 - 4(m - 2)m$
$= 4m + 1$

For the roots being rational, D must be a perfect square.

$\Rightarrow$ $4m + 1 = k^2$ for some $k \in I$

$\Rightarrow$ $m = \dfrac{(k-1)(k+1)}{4}$, clearly k must be odd.

Let $k = 2n + 1$

$\therefore$ $m = \dfrac{2n(2n+2)}{4} = n(n + 1), n \in I$

Hence, the correct answer is (3).

Example 4 Total number of values of a so that $x^2 - x - a = 0$ has integral roots, where $a \in N$ and $6 \le a \le 100$, is

(1) 2 (2) 4
(3) 8 (4) 6

Solution

$$x^2 - x - a = 0, D = 1 + 4a = \text{odd}$$

D must be perfect square of some odd integer.

Let $D = (2\lambda + 1)^2$

$\Rightarrow$ $1 + 4a = 1 + 4\lambda^2 + 4\lambda$

$\Rightarrow$ $a = \lambda(\lambda + 1)$

Since $a \in [6, 100]$,

$\Rightarrow$ $a = 6, 12, 20, 30, 42, 56, 72, 90$

Thus, a can attain 8 different values.

Hence, the correct answer is (3).

Example 5 Number of values of x satisfying the pair of quadratic equations $x^2 - px + 20 = 0$ and $x^2 - 20x + p = 0$ for some $p \in R$, is

(1) 1 (2) 2
(3) 3 (4) 4

Solution

Given equations are

$$x^2 - px + 20 = 0 \text{ and } x^2 - 20x + p = 0$$

If $p = 20$, both the quadratic equations are identical. Hence, $x = 10 + 4\sqrt{5}$ or $x = 10 - 4\sqrt{5}$ satisfy both the equations.

If $p \ne 20$ then $x^2 - px + 20 = x^2 - 20x + p$

$\Rightarrow$ $(20 - p)x + (20 - p) = 0$

$\Rightarrow$ $x = -1$ and $p = -21$

Hence, there are 3 values of x.

$$x = (10 + 4\sqrt{5}), (10 - 4\sqrt{5}), -1$$

Hence the correct answer is (3)

Example 6 Suppose $a, b, c \in I$ such that greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $(x + 1)$ and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ is $(x^3 - 4x^2 + x + 6)$. The value of $(a + b + c)$ is equal to

(1) -6 (2) -4
(3) -2 (4) 0

Solution

According to the question, we have

$$x^2 + ax + b \equiv (x + 1)(x + b)$$

Putting $x = 1$, we get

$b + 1 = a$ ....(1)

Also, $x^2 + bx + c \equiv (x + 1)(x + c)$

Putting $x = 1$, we get

$c + 1 = b$

or $b + 1 = c + 2$ ....(2)

Hence, $b + 1 = a = c + 2$

Also, $(x + 1)(x + b)(x + c) \equiv x^3 - 4x^2 + x + 6$

$x^3 + (1 + b + c)x^2 + (b + bc + c)x + bc$

$\equiv x^3 - 4x^2 + x + 6$

Comparing coefficients, we get

$1 + b + c = -4,$

$b + bc + c = 1,$

$bc = 6$

Solving, we get $c = -3$; $b = -2$ and $a = -1$

$\Rightarrow a + b + c = -6$

Hence, the correct answer is (1).

Example 7 The number of values of k for which $[x^2 - (k-2)x + k^2][x^2 + kx + (2k - 1)]$ is a perfect square is

(1) 2 (2) 1

(3) 0 (4) None of these

Solution

For given situation, $x^2 - (k-2)x + k^2 = 0$ and $x^2 + kx + 2k - 1 = 0$ should have both roots common or each should have equal roots.

If both roots are common then $\frac{1}{1} = \frac{-(k-2)}{k} = \frac{k^2}{2k-1}$

$\Rightarrow k = -k + 2$ and $2k - 1 = k^2 \Rightarrow k = 1$

If both equations have equal roots, then

$(k-2)^2 - 4k^2 = 0$ and $k^2 - 4(2k - 1) = 0$

$\Rightarrow (3k - 2)(-k - 2) = 0$ and $k^2 - 8k + 4 = 0,$

We get no common value from this case.

$\therefore k = 1$, is the only possible value.

Hence, the correct answer is (2).

Example 8 Let $f(x) = Ax^2 + Bx + C$ where A, B and C are real numbers. If $f(x)$ is an integer whenever x is an integer, then which of the following numbers is not necessarily integer?

(1) $2A$ (2) $A + B$

(3) C (4) none of these

Solution

Let us consider the integral values of x as 0, 1, –1. Then $f(0)$, $f(1)$ and $f(-1)$ are all integers.

So, C, $A + B + C$ and $A - B + C$ are all integers.

Since C is integer, $A + B$ and $A - B$ are also integers.

$2A = (A + B) + (A - B)$

So, $2A$, $A + B$ and C are all integers.

Hence, the correct answer is (4).

Example 9 The number of integral values of a for which the quadratic equation $(x + a)(x + 1991) + 1 = 0$ has integral roots are

(1) 3 (2) 0

(3) 1 (4) 2

Solution

$(x + a)(x + 1991) + 1 = 0$

$\Rightarrow (x + a)(x + 1991) = -1$

$\Rightarrow x + a = 1$ and $x + 1991 = -1$

$\Rightarrow a = 1993$

or $x + a = -1$ and $x + 1991 = 1 \Rightarrow a = 1989$

Hence, the correct answer is (4).

Example 10 Let α and β be the roots of the equation $x^2 - 2x + 3 = 0$. Then the equation whose roots are $P = \alpha^3 - 3\alpha^2 + 5\alpha - 2$ and $Q = \beta^3 - \beta^2 + \beta + 5$ is

(1) $x^2 + 3x + 2 = 0$ (2) $x^2 - 3x - 2 = 0$

(3) $x^2 - 3x + 2 = 0$ (4) none of these

Solution

It is given that α and β are the roots of equation $x^2 - 2x + 3 = 0$.

$\Rightarrow \alpha^2 - 2\alpha + 3 = 0$ ...(1)

and $\beta^2 - 2\beta + 3 = 0$ ...(2)

$\therefore \alpha^2 = 2\alpha - 3$ or $\alpha^3 = 2\alpha^2 - 3\alpha$

$\therefore P = (2\alpha^2 - 3\alpha) - 3\alpha^2 + 5\alpha - 2$

$= -\alpha^2 + 2\alpha - 2$

$= 3 - 2$ [Using (1)]

$= 1$

Similarly, we get $Q = 2$.

$\therefore$ Sum of roots = 3

and product of roots = 2

So, required equation is $x^2 - 3x + 2 = 0$.

Hence, the correct answer is (3).

Example 11 If $a(p + q)^2 + 2bpq + c = 0$ and $a(p + r)^2 + 2bpr + c = 0$, $(a \neq 0)$ then

(1) $qr = p^2$ (2) $qr = p^2 + \frac{c}{a}$

(3) $qr = -p^2$ (4) none of these

Solution

Given $a(p + q)^2 + 2bpq + c = 0$

and $a(p + r)^2 + 2bpr + c = 0$

So, q and r satisfy the equation.

$a(p + x)^2 + 2bpx + c = 0$

Thus, q and r are the roots of $a(p + x)^2 + 2bpx + c = 0$

or $ax^2 + 2(ap + bp)x + c + ap^2 = 0$

$\therefore \quad qr = \text{product of roots} = \dfrac{c + ap^2}{a} = p^2 + \dfrac{c}{a}$

Hence, the correct answer is (2).

Example 12 Let α and β be the roots of $x^2 + px + q = 0$ and also of $x^{2n} + p^n x^n + q^n = 0$. If $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ are the roots of $x^n + 1 + (x + 1)^n = 0$, then n ($\in N$)

(1) must be an odd integer (2) may be any integer
(3) must be an even integer (4) can't say anything

Solution

We have $\alpha + \beta = -p$ and $\alpha\beta = q$ ...(i)

Since α and β are the roots of $x^{2n} + p^n x^n + q^n = 0$,

we have $\alpha^{2n} + p^n\alpha^n + q^n = 0$ and $\beta^{2n} + p^n\beta^n + q^n = 0$.

Subtracting the above relations, we get

$(\alpha^{2n} - \beta^{2n}) + p^n(\alpha^n - \beta^n) = 0$

$\therefore \quad \alpha^n + \beta^n = -p^n$ ...(ii)

α/β or β/α is a root of $x^n + 1 + (x + 1)^n = 0$, then

$(\alpha/\beta)^n + 1 + [(\alpha/\beta) + 1]^n = 0$

or $\quad (\alpha^n + \beta^n) + (\alpha + \beta)^n = 0$

or $\quad -p^n + (-p)^n = 0$, [Using (i) and (ii)]

Above is possible only when n is even.

Hence, the correct answer is (3).

Example 13 If the equation $2x^2 + 4xy + 7y^2 - 12x - 2y + t = 0$, where t is a parameter, has exactly one real solution of the form (x, y), then the sum $(x + y)$ is equal to

(1) 3 (2) 5
(3) – 5 (4) – 3

Solution

$2x^2 + 4x(y - 3) + 7y^2 - 2y + t = 0$

$D = 0$ (For one solution)

$\Rightarrow \quad 16(y - 3)^2 - 8(7y^2 - 2y + t) = 0$

$\Rightarrow \quad 2(y - 3)^2 - (7y^2 - 2y + t) = 0$

$\Rightarrow \quad -5y^2 - 10y + 18 - t = 0$

$\Rightarrow \quad 5y^2 + 10y + t - 18 = 0$

Again, $D = 0$ (for one solution)

$\Rightarrow \quad 100 - 20(t - 18) = 0$

$\Rightarrow \quad 5 - t + 18 = 0$

$\therefore \quad t = 23$

For $\quad t = 23$,

$5y^2 + 10y + 5 = 0$

or $\quad (y + 1)^2 = 0$

$\Rightarrow \quad y = -1$

For $\quad y = -1$,

$2x^2 - 16x + 32 = 0$

or $\quad x^2 - 8x + 16 = 0$

$\Rightarrow \quad x = 4$

$\therefore \quad x + y = 3$

Hence, the correct answer is (1).

Example 14 Let $P(x) = x^3 - 8x^2 + cx + d$ be a polynomial with real coefficients and with all its zeros being distinct positive integers. Then which of the following is not the possible value of c?

(1) 17 (2) 19
(3) 12 (4) none of these

Solution

We have $x_1 + x_2 + x_3 = 8$

$x_1 x_2 x_3 = d$

$x_1x_2 + x_2x_3 + x_3x_1 = c$

Possible zeroes are 1, 2, 5 or 1, 3, 4.

$\therefore \quad d = 10$ or $d = 12$

$\Rightarrow \quad c = 2 + 10 + 5 = 17$ or $c = 3 + 12 + 4 = 19$

Hence, $d = 10$ and $c = 17$

or $\quad d = 12$ and $c = 19$

Hence, the correct answer is (3).

Example 15 For equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ if two of its roots are equal in magnitude but opposite in sign, then other two roots are

(1) real and negative (2) real and positive
(3) imaginary (4) none of these

Solution

Let the root be $\alpha, \beta, \gamma, \delta$

Let $\quad \alpha + \beta = 0$ but $\alpha + \beta + \gamma + \delta = 2 \;\therefore\; \gamma + \delta = 2$

Let $\quad \alpha\beta = p$ and $\gamma\delta = q$.

Given equation is equivalent to:

$(x^2 + p)(x^2 - 2x + q) = 0$

Comparing the coefficients, we get

$p + q = 4, -2p = 6, pq = -21$

$\therefore \quad p = -3, q = 7$ and they satisfy $pq = -21$

$\therefore \quad (x^2 - 3)(x^2 - 2x + 7) = 0$

Thus, roots are $\pm\sqrt{3}$ and $1 \pm i\sqrt{6}$.

Hence, the correct answer is (3).

Example 16 If the equation $x^2 + 9y^2 - 4x + 3 = 0$ is satisfied for real values of x and y, then which of the following is not true?

(1) $y \in [-1/3, 1/3]$ (2) $y \in [1, 3]$
(3) $x \in [1, 3]$ (4) none of these

Solution

Given equation is

$x^2 + 9y^2 - 4x + 3 = 0$ ...(1)

or $\quad x^2 - 4x + 9y^2 + 3 = 0$

Since x is real, we have

$$(-4)^2 - 4(9y^2 + 3) \geq 0$$

or $\quad 4 - 9y^2 - 3 \geq 0$

or $\quad y^2 \leq 1/9$

or $\quad -1/3 \leq y \leq 1/3$

Equation (1) can also be written as:

$$9y^2 + 0\,y + x^2 - 4x + 3 = 0 \qquad \ldots(2)$$

Since y is real, we have

$$0^2 - 4 \cdot 9(x^2 - 4x + 3) \geq 0$$

or $\quad x^2 - 4x + 3 \leq 0 \qquad \ldots(3)$

$\therefore \quad x \in [1, 3]$

Hence, the correct answer is (2).

Example 17 The set of values of a for which $ax^2 + (a-2)x - 2$ is negative for exactly two integral values of x, is

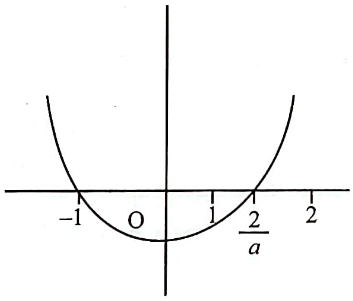
(1) $(0, 2)$ (2) $[1, 2)$
(3) $(1, 2]$ (4) $(0, 2]$

Solution

$$f(x) = ax^2 + (a-2)x - 2 = (ax - 2)(x + 1)$$

$$f(0) = -2 \text{ and } f(-1) = 0$$

If a is negative then expression becomes negative for infinite values of x, therefore it must be positive.



Expression has to be negative for exactly two integral values of x.

So, $\quad \dfrac{2}{a} \leq 2$

$\Rightarrow \quad a \geq 1$ and $\dfrac{2}{a} > 1$

$\Rightarrow \quad a < 2$

$\therefore \quad a \in [1, 2)$

Hence, the correct answer is (2).

Example 18 If a_0, a_1, a_2 and a_3 are all positive, then $4a_0 x^3 + 3a_1 x^2 + 2a_2 x + a_3 = 0$ has at least one root in $(-1, 0)$ for

(1) $4a_0 + 2a_2 > 3a_1 + a_3$
(2) $4a_0 + 2a_2 < 3a_1 + a_3$
(3) $4a_0 + 2a_2 = 3a_1 + a_3$ and $a_0 + a_2 < a_1 + a_3$
(4) none of these

Solution

$P(x) = 4a_0 x^3 + 3a_1 x^2 + 2a_2 x + a_3$ is a polynomial, so it is continuous for all x.

$P(x) = 0$ has a root in $(-1, 0)$.

$\Rightarrow \quad P(-1)P(0) < 0$

Now, $\quad P(0) = a_3 > 0$

$\Rightarrow \quad P(-1) = -4a_0 + 3a_1 - 2a_2 + a_3 < 0$

$\Rightarrow \quad 4a_0 + 2a_2 > 3a_1 + a_3$

Hence, the correct answer is (1).

NCERT LEVEL EXERCISE

1. Solve the equation $x^2 + 4 = 0$.
2. Solve the equation $2x^2 + x + 2 = 0$.
3. Solve the equation $x^2 + 2x + 8 = 0$.
4. Solve the equation $-x^2 + x - 5 = 0$.
5. Solve the equation $x^2 - x + 7 = 0$.
6. Solve the equation $\sqrt{2}\,x^2 + x + \sqrt{8} = 0$.
7. Solve the equation $\sqrt{5}\,x^2 - 2x + 2\sqrt{5} = 0$.
8. Solve the equation $x^2 + \frac{x}{\sqrt{7}} + 1 = 0$.

STATE LEVEL EXERCISES

Single Correct Answer Type

1. The roots of the equation $x^{2/3} + x^{1/3} - 2 = 0$ are
(1) 1, 4 (2) 1, – 4
(3) 1, – 8 (4) 1, 8

2. The solution set of the equation $x^{\log_x (1-x)^2} = 9$ is
(1) {– 2, 4} (2) {4}
(3) {0, – 2, 4} (4) none of these

3. If a and b are the odd integers, then the roots of the equation $2ax^2 + (2a + b)x + b = 0, a \neq 0$, will be
(1) rational (2) irrational
(3) non-real (4) equal

4. The value of $2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots\infty}}}$ is
(1) $2 + \sqrt{2}$ (2) $1 + \sqrt{2}$
(3) 3 (4) 3.2

5. The curve $y = (\lambda + 1)x^2 + 2$ intersects the curve $y = \lambda x + 3$ in exactly one point, if λ equals
(1) {–2, 2} (2) {1}
(3) {–2} (4) {2}

6. If $a, b, c \in R^+$ and $2b = a + c$ then the roots of $ax^2 + 2bx + c = 0$ are
(1) real and distinct (2) real and equal
(3) imaginary (4) none of these

7. If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, then a, b, c are in
(1) A.P. (2) H.P.
(3) G.P. (4) none of these

8. If quadratic equation $x^2 - 4x - \log_3 a = 0$ has real roots then the minimum value of a is
(1) 81 (2) $\frac{1}{81}$
(3) $\frac{1}{64}$ (4) 9

9. If $a + b + c = 0$, then the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are
(1) real and equal (2) imaginary
(3) real and unequal (4) none of these

10. Let a, b and c be real numbers such that $4a + 2b + c = 0$ and $ab > 0$. Then the equation $ax^2 + bx + c = 0$ has
(1) complex roots (2) exactly one root
(3) real roots (4) none of these

11. The coefficient of x in the equation $x^2 + px + q = 0$ was wrongly written as 17 in place of 13 and the roots thus found are –2 and –15. The roots of the correct equation are
(1) –3, 10 (2) –3, –10
(3) 3, –10 (4) none of these

12. If the sum of the roots of the equation $x^2 + px + q = 0$ is three times their difference, then which one of the following is true?
(1) $9p^2 = 2q$ (2) $2q^2 = 9p$
(3) $2p^2 = 9q$ (4) $9q^2 = 2p$

13. If A.M. of the roots of a quadratic equation is 8/5 and A.M. of their reciprocals is 8/7, then the equation is
(1) $7x^2 - 16x + 8 = 0$ (2) $3x^2 - 12x + 7 = 0$
(3) $5x^2 - 16x + 7 = 0$ (4) $7x^2 - 16x + 5 = 0$

14. If $\sin\theta$ and $\cos\theta$ are the roots of $ax^2 + bx + c = 0$ then
(1) $a^2 = b^2 + 2ac$ (2) $b^2 = a^2 + 2ac$
(3) $a^2 = b^2 + 2ab$ (4) $b^2 = a^2 + 2ab$

15. If α and β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 8)$ is
(1) $x^2 - 5x + 6 = 0$ (2) $x^2 - 3x + 2 = 0$
(3) $x^2 + 4x - 1 = 0$ (4) $x^2 + 2x + 3 = 0$

16. If the roots of equation $x^2 + qx + p = 0$ are each 1 less than the roots of the equation $x^2 + px + q = 0$, then $(p + q)$ is equal to
(1) – 2 (2) – 4
(3) – 5 (4) – 6

17. If $(x - 2)$ is a common factor of the expressions $x^2 + ax + b$ and $x^2 + cx + d$, then $\frac{b-d}{c-a} =$
(1) – 2 (2) – 1
(3) 1 (4) 2

18. If x is real and $k = \frac{x^2 - x + 1}{x^2 + x + 1}$ then
(1) $k \leq 0$ (2) $1/3 \leq k \leq 3$
(3) $k \geq 5$ (4) none of these

19. Let $f(x)$ be a quadratic expression such that $f(-1) + f(2) = 0$. If one root of $f(x) = 0$ is 3 then the other root of $f(x) = 0$ lies in
(1) $(-\infty, -3)$ (2) $(-3, \infty)$
(3) (0, 5) (4) $(5, \infty)$

20. If $k \in (-\infty, -2) \cup (2, \infty)$, then the roots of the equation $x^2 + 2kx + 4 = 0$ are
(1) complex
(2) real and unequal
(3) real and equal
(4) one real and other imaginary

21. If the roots of the equation $qx^2 + px + q = 0$, where p, q are real, be complex, then the roots of the equation $x^2 - 4qx + p^2 = 0$ are
(1) real and unequal (2) real and equal
(3) imaginary (4) none of these

22. If α and β are roots of the equation $2x^2 - (p+1)x + (p-1) = 0$ and $\alpha - \beta = \alpha\beta$, then the value of p is
(1) 3 (2) -2
(3) 1 (4) 2

23. The roots of $(x-a)(x-a-1) + (x-a-1)(x-a-2) + (x-a)(x-a-2) = 0$, $a \in R$ are always
(1) equal (2) imaginary
(3) real and distinct (4) rational and equal

24. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where p and q are real, then $(p, q) =$
(1) $(4, -7)$ (2) $(4, 7)$
(3) $(-4, -7)$ (4) $(-4, 7)$

25. If α and β are the roots of the equations $x^2 + 6x + \lambda = 0$ and $3\alpha + 2\beta = -20$, then $\lambda =$
(1) 16 (2) -8
(3) -16 (4) 8

26. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then $\dfrac{\alpha}{a\beta + b} + \dfrac{\beta}{a\alpha + b} =$
(1) $2/c$ (2) $-2/a$
(3) $2/a$ (4) $2/b$

27. The equation $x^2 + ax - a^2 - 1 = 0$ will have roots of opposite signs if
(1) $a \in (-\infty, \infty)$ (2) $a \in [-1, 1]$
(3) $a \in (-\infty, -1) \cup [1, \infty)$ (4) none of these

28. Let $f(x) = x^2 + 4x + 1$. Then
(1) $f(x) > 0$ for all x (2) $f(x) > 1$ when $x \geq 0$
(3) $f(x) \geq 1$ when $x \leq -4$ (4) $f(x) = f(-x)$ for all x

29. $kx^2 + 2kx + \dfrac{1}{2} > 0, \forall x \in R$, then complete set of values of k is
(1) $\left(0, \dfrac{1}{2}\right)$ (2) $(-\infty, 0) \cup \left(\dfrac{1}{2}, \infty\right)$
(3) $\left(\dfrac{1}{2}, \infty\right)$ (4) $\left[0, \dfrac{1}{2}\right)$

30. The graph of a quadratic polynomial (parabola) opens downward, with y-intercept 10 and x-intercepts -1 and 5. If the point $P(8, k)$ lies on the graph of the parabola, then the value of k is
(1) -60 (2) -54
(3) -27 (4) -8

31. If a, b, c are real and $x^3 - 3b^2x + 2c^3$ is divisible by $x - a$ and $x - b$, then
(1) $a = -b = -c$ (2) $a = 2b = 2c$
(3) $a = b = c$ or $a = -2b = -2c$ (4) none of these

Archives

1. If both the roots of $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6k(2x^2 + 1) + px + 4x^2 - 2 = 0$ are common, then $2r - p$ is equal to
(1) -1 (2) 0
(3) 1 (4) 2 **[MNR 1983]**

2. If $x^2 - hx - 21 = 0$ and $x^2 - 3hx + 35 = 0$ $(h > 0)$ have a common root, then the value of h is equal to
(1) 1 (2) 2
(3) 3 (4) 4
[EAMCET 1986]

3. If $x^2 + 2x + 2xy + my - 3$ has two rational factors, then the value of m will be
(1) $-6, -2$ (2) $-6, 2$
(3) $6, -2$ (4) $6, 2$
[RPET 1990]

4. If the roots of $x^2 + x + a = 0$ exceed a, then
(1) $2 < a < 3$ (2) $a > 3$
(3) $-3 < a < 3$ (4) $a < -2$
[EAMCET 1994]

5. A quadratic equation whose product of roots x_1 and x_2 is equal to 4 and satisfying the relation $\dfrac{x_1}{x_1 - 1} + \dfrac{x_2}{x_2 - 1} = 2$, is
(1) $x^2 - 2x + 4 = 0$ (2) $x^2 + 2x + 4 = 0$
(3) $x^2 + 4x + 4 = 0$ (4) $x^2 - 4x + 4 = 0$
[Kurukshetra CEE 1998]

6. If sum of roots is -1 and sum of their reciprocals is $\dfrac{1}{6}$, then equation is
(1) $x^2 + x - 6 = 0$ (2) $x^2 - x + 6 = 0$
(3) $6x^2 + x + 1 = 0$ (4) $x^2 - 6x + 1 = 0$
[Karnataka CET 1998]

7. If x be real, the least value of $x^2 - 6x + 10$ is
(1) 1 (2) 2
(3) 3 (4) 10
[Kurukshetra CEE 1998]

8. The two roots of an equation $x^3 - 9x^2 + 14x + 24 = 0$ are in the ratio 3 : 2. The roots will be
(1) 6, 4, -1 (2) 6, 4, 1
(3) -6, 4, 1 (4) -6, -4, 1
[UPSEAT 1999]

9. $\{x \in R : |x - 2| = x^2\} =$
(1) $\{-1, 2\}$ (2) $\{1, 2\}$
(3) $\{-1, -2\}$ (4) $\{1, -2\}$
[EAMCET 2000]

10. If $\sin A$, $\sin B$, $\cos A$ are in G.P., then roots of $x^2 + 2x \cot B + 1 = 0$ are always
(1) real (2) imaginary
(3) greater than 1 (4) equal
[Orissa JEE 2005]

11. If α and β are the roots of the equation $ax^2 + bx + c = 0$ then the sum of the roots of the equation $a^2x^2 + (b^2 - 2ac)x + b^2 - 4ac = 0$ is

(1) $-(\alpha^2 - \beta^2)$ (2) $(\alpha + \beta)^2 - 2\alpha\beta$
(3) $\alpha^2\beta + \beta\alpha^2 - 4\alpha\beta$ (4) $-(\alpha^2 + \beta^2)$

[EAMCET 2000]

12. If one root of the equation $x^2 + p + q = 0$ is $2 + \sqrt{3}$, then values of p and q are

(1) $-4, 1$ (2) $4, -1$
(3) $2, \sqrt{3}$ (4) $-2, -\sqrt{3}$

[UPSEAT 2002]

13. If the equation $x^2 + px + q = 0$ and $x^2 + qx + p = 0$, have a common root, then $p + q + 1 =$

(1) 0 (2) 1
(3) 2 (4) -1

[Orissa JEE 2002]

14. The roots of the equation $x^4 - 2x^3 + x = 380$ are

(1) $5, -4, \frac{1 \pm 5\sqrt{-3}}{2}$ (2) $-5, 4, -\frac{1 \pm 5\sqrt{-3}}{2}$
(3) $5, 4, \frac{-1 \pm 5\sqrt{-3}}{2}$ (4) $-5, -4, \frac{1 \pm 5\sqrt{-3}}{2}$

[UPSEAT 2004]

15. The roots of the equation $x^{2/3} + x^{1/3} - 2 = 0$ are

(1) 1, 4 (2) 1, –4
(3) 1, –8 (4) 1, 8

[UPSEAT 2004]

16. The real roots of the equation $x^2 + 5|x| + 4 = 0$ are

(1) $-1, 4$ (2) $1, 4$
(3) $-4, 4$ (4) None of these

[UPSEAT 1993, Orissa JEE 2004]

17. $x^2 + x + 1 + 2k(x^2 - x - 1) = 0$ is a perfect square for how many values of k?

(1) 2 (2) 0
(3) 1 (4) 3

[Orissa JEE 2004]

18. A quadratic equation with integral coefficients has two different prime numbers as its roots. If the sum of the coefficients of the equation is prime then the sum of the roots is

(1) 2 (2) 5
(3) 7 (4) 11

[Orissa JEE 2005]

19. If α, β and γ are roots of the equation $x^3 - 8x + 8 = 0$ then the value of $\alpha^2 + \beta^2 + \gamma^2$ and $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$ are, respectively,

(1) 16 and 0 (2) -16 and 0
(3) 16 and 8 (4) 0 and -16

(KCET 2006)

20. If $a, -a$ and b are the roots of the equation $x^3 - 5x^2 - x + 5 = 0$ then b is root of the equation

(1) $x^2 - 3x - 10 = 0$ (2) $x^2 + 5x - 30 = 0$
(3) $x^2 + 3x - 20 = 0$ (4) $x^2 - 5x + 10 = 0$

(KCET 2010)

21. If $a > 0$ and $b^2 - 4ac = 0$, then the curve $y = ax^2 + bx + c$

(1) cuts the x-axis
(2) touches the x-axis and lies below it
(3) lies entirely above the x-axis
(4) touches the x-axis and lies above it

(EAMCET 2011)

22. If $\tan A$ and $\tan B$ are the roots of the quadratic equation $x^2 - px + q = 0$, then $\sin^2(A + B) =$

(1) $\frac{p^2}{p^2 + q^2}$ (2) $\frac{p^2}{(p + q)^2}$
(3) $1 - \frac{p}{(1 - q)^2}$ (4) $\frac{p^2}{p^2 + (1 - q)^2}$

(EAMCET 2011)

23. The value of a for which the equations $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$ have a common root is

(1) -2 (2) -1
(3) 1 (4) 2

(EAMCET 2011)

24. If x is real, then the value of $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$ lies in the interval

(1) $\left[\frac{1}{3}, 3\right]$ (2) $\left[\frac{1}{5}, 5\right]$
(3) $\left[\frac{1}{6}, 6\right]$ (4) $\left[\frac{1}{7}, 7\right]$

(EAMCET 2011)

25. In a triangle PQR, $\angle R = \frac{\pi}{4}$. If $\tan\left(\frac{P}{3}\right)$ and $\tan\left(\frac{Q}{3}\right)$ are the roots of the equation $ax^2 + bx + c = 0$, then

(1) $a + b = c$ (2) $b + c = 0$
(3) $a + c = b$ (4) $b = c$

(EAMCET 2012)

26. If α, β and γ are roots of the equation $x^3 + 4x + 2 = 0$, then the value of $\alpha^3 + \beta^3 + \gamma^3$ is

(1) 2 (2) 6
(3) –2 (4) –6

[KCET 2012]

27. If x_1 and x_2 are the real roots of the equation $x^2 - kx + c = 0$ then the distance between the points $A(x_1, 0)$ and $B(x_2, 0)$ is

(1) $\sqrt{k^2 + 4c}$ (2) $\sqrt{k^2 - c}$
(3) $\sqrt{c - k^2}$ (4) $\sqrt{k^2 - 4c}$

(EAMCET 2014)

28. If p and q are distinct prime numbers and if the equation $x^2 - px + q = 0$ has positive integers as its roots then the roots of the equation are

(1) 1, –1 (2) 2, 3
(3) 1, 2 (4) 3, 1

(WBJEE 2014)

29. The cubic equation whose roots are the squares of the roots of $x^3 - 2x^2 + 10x - 8 = 0$ is

(1) $x^3 + 16x^2 + 68x - 64 = 0$
(2) $x^3 + 8x^2 + 68x - 64 = 0$
(3) $x^3 + 16x^2 - 68x - 64 = 0$
(4) $x^3 - 16x^2 + 68x - 64 = 0$ **(EAMCET 2014)**

30. If a, b, c are distinct and the roots of $(b - c)x^2 + (c - a)x + (a - b) = 0$ are equal, then a, b, c are in

(1) arithmetic progression
(2) geometric progression
(3) harmonic progression
(4) arithmetic-geometric progression

(EAMCET 2015)

31. If the roots of $x^3 - kx^2 + 14x - 8 = 0$ are in geometric progression, then $k =$

(1) – 3 (2) 7
(3) 4 (4) 0

(EAMCET 2015)

32. If the harmonic mean of the roots of $\sqrt{2}x^2 - bx + (8 - 2\sqrt{5})$ is 4, then the value of $b =$

(1) 2 (2) 3
(3) $4 - \sqrt{5}$ (4) $4 + \sqrt{5}$

(EAMCET 2015)

33. If α and β are roots of the equation $x^2 + x + 1 = 0$, then $\alpha^2 + \beta^2$ is

(1) 1 (2) $\frac{-1-i\sqrt{3}}{2}$

(3) $\frac{-1+i\sqrt{3}}{2}$ (4) –1 **(KCET 2019)**

34. The largest interval containing x for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is

(1) $0 < x < 1$ (2) $-4 < x < 2$
(3) $-\infty < x < \infty$ (4) $-2^{10} < x < 2^{10}$

(EAPCET 2022)

35. **Statement (I):** The set of solution of $|x|^2 - 4|x| + 3 < 0$ is the interval (–3, 3).

Statement (II): If $x < 3$ or $x > 5$ then $x^2 - 8x + 15 > 0$.

Which of the above statements is (are) true?

(1) Statement I is true, but Statement II is false
(2) Statement II is true, but Statement I is false
(3) Both Statement I and Statement II are true
(4) Both Statement I and Statement II are false

(EAMCET 2022)

36. If the roots of the equation $6x^3 - 11x^2 + 6x - 1 = 0$ are in harmonic progression, then the roots of $x^3 - 6x^2 + 11x - 6 = 0$ will be in

(1) Geometric progression
(2) Arithmetic progression
(3) Harmonic progression
(4) Arithmetico–Geometric progression

(EAPCET 2023)

37. The sum of all the real values of x satisfying the equation $(x^2 - 7x + 11)^{x^2 - 6x - 7} = 1$ is

(1) 14 (2) 20
(3) 13 (4) 16

(EAMCET 2023)

JEE LEVEL EXERCISES

Single Correct Answer Type

Polynomials

1. If x is real, then the maximum value of $5 + 4x - 4x^2$ will be equal to
(1) 5 (2) 6
(3) 1 (4) 2

2. If $x = 2 + 2^{2/3} + 2^{1/3}$, then $x^3 - 6x^2 + 6x =$
(1) 3 (2) 2
(3) 1 (4) None of these

3. If the graph of $f(x) = 2x^3 + ax^2 + bx$; $a, b \in N$ cuts the x-axis at three distinct points, then the minimum value of $(a + b)$ is
(1) 2 (2) 3
(3) 4 (4) 5

4. If $f(x) = ax^2 + bx + c$ and $f(-1) \geq -4$, $f(1) \leq 0$ and $f(3) \geq 5$, then the least value of a is
(1) 1/4 (2) 1/8
(3) 1/3 (4) – 1/3

5. Number of positive integers x for which $f(x) = x^3 - 8x^2 + 20x - 13$, is a prime number, is
(1) 1 (2) 2
(3) 3 (4) 4

6. Let $P(x) = x^2 + bx + c$, where b and c are integers. If $P(x)$ is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, then the value of $P(1)$ is
(1) 2 (2) 3
(3) 4 (4) 5

7. If $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, then $a(\alpha x + 1)(\beta x + 1)$ is equal to
(1) $ax^2 + bx + c$ (2) $cx^2 - bx + a$
(3) $cx^2 - bx - a$ (4) $cx^2 + bx + a$

8. If $x^2 + ax + 1$ is a factor of $ax^3 + bx + c$ then
(1) $b + a + a^2 = 0, a = c$
(2) $b - a + a^2 = 0, a = c$
(3) $b + a - a^2 = 0, a = 0$
(4) $b - a + a^3 = 0, a^2 + c = 0$

9. If α, β, γ and σ are the zeros of the equation $x^4 + 4x^3 - 6x^2 + 7x - 9 = 0$, then the value of $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$ is
(1) 9 (2) 11
(3) 13 (4) 5

10. If α, β and γ are the zeros of the equation $x^3 + px^2 + qx + r = 0$, then $(1 - \alpha^2)(1 - \beta^2)(1 - \gamma^2)$ is equal to
(1) $(1 + p)^2 - (q + r)^2$ (2) $(1 + q)^2 - (p + r)^2$
(3) $(1 - q)^2 - (p + r)^2$ (4) $(1 + r)^2 - (p + q)^2$

11. The number of values of the pair (a, b) for which $a(x + 1)^2 + b(x^2 - 3x - 2) + x + 1 = 0 \ \forall x \in R$ is
(1) 0 (2) 1
(3) 2 (4) infinite

12. If a, b, c are three distinct real numbers then the equation $\frac{(x-b)(x-c)}{(a-b)(a-c)}a^2 + \frac{(x-c)(x-a)}{(b-c)(b-a)}b^2 + \frac{(x-a)(x-b)}{(c-a)(c-b)}c^2 - x^2 = 0$ has
(1) exactly one root (2) exactly two roots
(3) no root (4) none of these

Quadratic Equations

13 If a, b, c are in A.P. and one root of the equation $ax^2 + bx + c = 0$ is 2 then the other root is
(1) 3/4 (2) – 3/4
(3) – 5/2 (4) –5/4

14. The product of real roots of the equation $|x|^{\frac{6}{5}} - 26|x|^{\frac{3}{5}} - 27 = 0$ is
(1) -3^{10} (2) -3^{12}
(3) $-3^{\frac{12}{5}}$ (4) $-3^{\frac{21}{5}}$

15. Number of real solutions of $\sqrt{2x-4} - \sqrt{x+5} = 1$ is
(1) 0 (2) 1
(3) 2 (4) infinite

16. Sum of all the real values of x which satisfy the equation $\frac{\sqrt{2-x}}{\sqrt{2+x}} = \frac{2-x}{2+x}$ is
(1) 2 (2) 0
(3) 7.5 (4) 11.5

17. Sum of the non-real roots of $(x^2 + x - 2)(x^2 + x - 3) = 12$ is
(1) –1 (2) 1
(3) –6 (4) 6

18. Product of roots of the equation $\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$ is
(1) 1 (2) – 1
(3) 2 (4) –2

19. The number of irrational roots of the equation $\frac{4x}{x^2 + x + 3} + \frac{5x}{x^2 - 5x + 3} = -\frac{3}{2}$ is
(1) 3 (2) 0
(3) 1 (4) 2

20. Number of real solutions of the equation $\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}=x$ is

(1) 1 (2) 2
(3) 0 (4) infinite

21. Total number of integral values of a so that $x^2-(a+1)x+a-1=0$ has integral roots, is equal to

(1) 1 (2) 2
(3) 4 (4) none of these

22. Consider the equation $x^2+2x-n=0$, where $n \in N$ and $n \in [5, 100]$. Total number of different values of n so that the given equation has integral roots is

(1) 8 (2) 3
(3) 6 (4) 4

Nature of Roots of Quadratic Equations

23. If the roots of the equation $x^2-2cx+ab=0$ be real and unequal, the roots of the equation $x^2-2(a+b)x+(a^2+b^2+2c^2)=0$ are

(1) real and distinct (2) real and equal
(3) real (4) imaginary

24. If $a \in (-1, 1)$, then roots of the quadratic equation $(a-1)x^2+ax+\sqrt{1-a^2}=0$ are

(1) real (2) imaginary
(3) both equal (4) none of these

25. If one root of the equation $x^2+px+12=0$ is 4, while the equation $x^2+px+q=0$ has equal roots, then the value of q is

(1) 4 (2) 49/4
(3) 4/49 (4) None of these

26. If α, β are real and α^2, β^2 are the roots of the equation $a^2x^2+x+1-a^2=0$ $(a>1)$, then $\beta^2=$

(1) a^2 (2) $1-\frac{1}{a^2}$
(3) $1-a^2$ (4) $1+a^2$

27. Let α, β be the roots of $x^2+bx+1=0$. Then the equation whose roots are $-\left(\alpha+\frac{1}{\beta}\right)$ and $-\left(\beta+\frac{1}{\alpha}\right)$ is

(1) $x^2-2bx+4=0$ (2) $x^2-bx+1=0$
(3) $x^2=0$ (4) $x^2+2bx+4=0$

28. If α and β are the roots of the equation $2x^2+2(a+b)x+a^2+b^2=0$, then the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$ is

(1) $x^2-4abx-(a^2-b^2)^2=0$
(2) $x^2-2abx-(a^2-b^2)^2=0$
(3) $x^2-4abx+(a^2-b^2)^2=0$
(4) none of these

29. If the expression $x^2-(5m-2)x+(4m^2+10m+25)$ can be expressed as a perfect square, then $m=$

(1) $\frac{8}{3}$ or 4 (2) $-\frac{8}{3}$ or 4
(3) $\frac{4}{3}$ or 8 (4) $-\frac{4}{3}$ or 8

30. The roots of the equation $(a^4+b^4)x^2+4abcdx+(c^4+d^4)=0$

(1) cannot be different, if real (2) are always real
(3) are always imaginary (4) none of these

31. $F(x)=Ax^2+Bx+C$ and $f(x)=ax^2+bx+c$ are quadratic functions with $F(x) \neq f(x)$. Which of the following is true about the number of solutions to $F(x)-f(x)=0$?

(I) It is possible that there is no real solution.
(II) It cannot have more than 2 solutions.
(III) If it has one real solution then $A=a$.

(1) I and II (2) II and III
(3) III and I (4) I, II and III

32. Complete set of values of a such that $\frac{x^2-x}{1-ax}$ attains all real values is

(1) $[1, 4]$ (2) $(0, 4]$
(3) $[1, \infty)$ (4) none of these

33. The set of values of a for which the inequation $(a-1)x^2-(a+1)x+a-1 \geq 0$ is true for all $x \geq 2$ is

(1) $\left(1, \frac{7}{3}\right]$ (2) $(-\infty, 1)$
(3) $\left[\frac{7}{3}, \infty\right)$ (4) none of these

34. For real values of x, the range of $\frac{x^2+2x+1}{x^2+2x-1}$ is

(1) $(-\infty, 0) \cup (1, \infty)$ (2) $\left[\frac{1}{2}, 2\right]$
(3) $\left(-\infty, \frac{-2}{9}\right] \cup (1, \infty)$ (4) $(-\infty, -6] \cup (-2, \infty)$

35. The set of real values of a for which the equation $\frac{2a^2+x^2}{a^3-x^3}-\frac{2x}{ax+a^2+x^2}+\frac{1}{x-a}=0$ has a unique solution is

(1) $(-\infty, 1)$ (2) $(-1, \infty)$
(3) $(-1, 1)$ (4) $R-\{0\}$

36. Assume that p is a real number. In order for $\sqrt[3]{x+3p+1}-\sqrt[3]{x}=1$ to have real solutions, it is necessary that

(1) $p \geq 1/4$ (2) $p \geq -1/4$
(3) $p \geq 1/3$ (4) $p \geq -1/3$

37. If $(ax^2+c)y+(a'x^2+c')=0$ such that x is a rational function of y and ac is negative, then

(1) $ac'+a'c=0$ (2) $\frac{a}{a'}=\frac{c}{c'}$

(3) $a^2+c^2=a'^2+c'^2$ (4) $aa'+cc'=1$

38. If $a, b, c, d \in R$ then the equation $(x^2+ax-3b)(x^2-cx+b)(x^2-dx+2b)=0$ has

(1) six real roots (2) at least two real roots

(3) four real roots (4) three real roots

39. If the left hand side of the equation $x^2-y^2+x-3y+\sec\theta=0$ can be factorized into two linear factors then the value of θ is

(1) $\pi/3$ (2) $7\pi/6$

(3) $4\pi/3$ (4) $5\pi/6$

Relation between Roots and Coefficients of Polynomial Equations

40. If the roots of the equation $ax^2-4x+a^2=0$ are imaginary and the sum of the roots is equal to their product, then a is

(1) –2 (2) 4

(3) 2 (4) none of these

41. If α, β are the roots of $x^2-3x+5=0$ and γ, δ are the roots of $x^2+5x-3=0$ then the equation whose roots are $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$ is

(1) $x^2-15x-158=0$ (2) $x^2+15x-158=0$

(3) $x^2-15x+158=0$ (4) $x^2+15x+158=0$

42. If the absolute value of the difference of roots of the equation $x^2+px+1=0$ exceeds $\sqrt{3}p$ then

(1) $-1<p<4$ (2) $0\le p<4$

(3) $p<-1$ or $p>4$ (4) $p>4$

43. All possible values of the parameter a such that the roots α and β of the equation $2x^2+6x+a=0$ satisfy the inequality $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}<2$ are

(1) $a>0$ (2) $a<9/2$

(3) $a<0$ or $a>9/2$ (4) none of these

44. The value of a for which the sum of the squares of the roots of the equation $x^2-(a-2)x-a-1=0$ assumes the least value is

(1) 2 (2) 3

(3) 0 (4) 1

45. If the roots of the equation $ax^2+bx+c=0$ are reciprocal to each other, then

(1) $a+c=0$ (2) $b+c=0$

(3) $a-c=0$ (4) $b-c=0$

46. If α, β are the roots of $x^2+px+1=0$ and γ, δ are the roots of $x^2+qx+1=0$, then $q^2-p^2=$

(1) $(\alpha+\gamma)(\beta+\gamma)(\alpha-\delta)(\beta+\delta)$

(2) $(\alpha+\gamma)(\beta+\gamma)(\alpha+\delta)(\beta+\delta)$

(3) $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$

(4) none of these

47. If the equation $f(x)=(a+b-c)x^2+(b+c-a)x+(c+a-b)=0$ has only one root equal to 1, then the value of $\frac{3b+a}{c}$ is

(1) 0 (2) –1

(3) 1 (4) 2

48. If α, β are the roots of $x^2+px+q=0$ and γ, δ are the roots of $x^2+px-r=0$ then $\frac{\alpha-\gamma}{\beta-\gamma}\cdot\frac{\alpha-\delta}{\beta-\delta}$ is equal to

(1) 1 (2) –1

(3) $\frac{q+r}{p+r}$ (4) $\frac{q-r}{p-r}$

49. If α and β are the roots of the equation $x^2+px-\frac{1}{2p^2}=0$, where $p\in R$, then the minimum value of $\alpha^4+\beta^4$ is

(1) $2\sqrt{2}$ (2) $2-\sqrt{2}$

(3) 2 (4) $2+\sqrt{2}$

50. If α, β are the roots of $ax^2+bx+c=0$ and $\alpha+h, \beta+h$ are the roots of $px^2+qx+r=0$, then $h=$

(1) $-\frac{1}{2}\left(\frac{a}{b}-\frac{p}{q}\right)$ (2) $\left(\frac{b}{a}-\frac{q}{p}\right)$

(3) $\frac{1}{2}\left(\frac{b}{a}-\frac{q}{p}\right)$ (4) none of these

51. If the roots of the equation $ax^2-bx+c=0$ are α and β then the roots of the equation $b^2cx^2-ab^2x+a^3=0$ are

(1) $\frac{1}{\alpha^3+\alpha\beta}, \frac{1}{\beta^3+\alpha\beta}$ (2) $\frac{1}{\alpha^2+\alpha\beta}, \frac{1}{\beta^2+\alpha\beta}$

(3) $\frac{1}{\alpha^4+\alpha\beta}, \frac{1}{\beta^4+\alpha\beta}$ (4) None of these

52. If α, β are the non-zero roots of $ax^2+bx+c=0$ and α^2, β^2 are the roots of $a^2x^2+b^2x+c^2=0$ then a, b, c are in

(1) G.P. (2) H.P.

(3) A.P. (4) none of these

53. Let $f(x)=ax^2+bx+c$, $g(x)=ax^2+qx+r$, where $a, b, c, q, r \in R$ and $a<0$. If α, β are the roots of $f(x)=0$ and $\alpha+\delta, \beta+\delta$ are the roots of $g(x)=0$, then

(1) $f_{max}>g_{max}$

(2) $f_{max}<g_{max}$

(3) $f_{max}=g_{max}$

(4) can't say anything about relation between f_{max} and g_{max}

54. If the roots of the equation $ax^2 + bx + c = 0$ are of the from $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, then $(a + b + c)^2$ is equal to

(1) $2b^2 - ac$ (2) $a^2 + b^2 + c^2$
(3) $b^2 - 4ac$ (4) $b^2 - 2ac$

55. If α and β are the roots of $ax^2 + c = bx$, then the equation $(a + cy)^2 = b^2y$ in y has the roots

(1) $\alpha\beta^{-1}, \alpha^{-1}\beta$ (2) α^{-2}, β^{-2}
(3) α^{-1}, β^{-1} (4) α^2, β^2

56. If one root of the equation $ax^2 + bx + c = 0$ is the square of the other, then $a(c - b)^3 = ck$, where k is

(1) $a^3 - b^3$ (2) $a^3 + b^3$
(3) $(a - b)^3$ (4) None of these

Higher Degree Equations

57. If the roots of $x^3 - 42x^2 + 336x - 512 = 0$, are in increasing geometric progression, then its common ratio is

(1) 2 (2) 3
(3) 4 (4) 6

58. If the roots of $x^4 + qx^2 + kx + 225 = 0$ are in arithmetic progression, then the value of q is

(1) 15 (2) 25
(3) 35 (4) –50

59. If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ then the equation whose roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$ is

(1) $d^2x^3 - bdx^2 + acx - a^2 = 0$
(2) $d^2x^3 + acx^2 - bdx - a^2 = 0$
(3) $d^2x^3 - acx^2 + bdx - a^2 = 0$
(4) $d^2x^3 + bdx^2 - acx - a^2 = 0$

60. If α, β and γ are the roots of $4x^3 + 8x^2 - x - 2 = 0$, then the value of $\frac{4(\alpha+1)(\beta+1)(\gamma+1)}{\alpha\beta\gamma}$ is equal to

(1) $\frac{3}{2}$ (2) 6
(3) $-\frac{3}{2}$ (4) –6

61. Let a, b, c be three distinct non-zero real numbers satisfying the system of following equations:

$$\frac{1}{a}+\frac{1}{a-1}+\frac{1}{a-2}=1;\ \frac{1}{b}+\frac{1}{b-1}+\frac{1}{b-2}=1;$$
$$\frac{1}{c}+\frac{1}{c-1}+\frac{1}{c-2}=1$$

Then $abc =$

(1) 1 (2) 2
(3) 3 (4) 4

Common Root(s) of Equations

62. The value of m for which one of the roots of $x^2 - 3x + 2m = 0$ is double of one of the roots of $x^2 - x + m = 0$ is

(1) –2 (2) 1
(3) 2 (4) none of these

63. If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root and a, b, c are non-zero real numbers then $\frac{a^3+b^3+c^3}{abc}$ is equal to

(1) 3 (2) 1
(3) 2 (4) None of these

64. If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, then $a : b : c$ is

(1) 1 : 2 : 9 (2) 1 : 3 : 9
(3) 3 : 2 : 8 (4) none of these

65. If the two equations $x^2 - cx + d = 0$ and $x^2 - ax + b = 0$ have one common root and the second equation has equal roots, then $2(b + d) =$

(1) ac (2) $-ac$
(3) 0 (4) $a + c$

66. If $x^2 + ax + 10 = 0$ and $x^2 + bx - 10 = 0$ have a common root, then $a^2 - b^2$ is equal to

(1) 40 (2) 10
(3) 20 (4) 30

67. If the equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common positive root then ordered pair (a, b) is

(1) (–6, –7) (2) (–7, –9)
(3) (–6,–8) (4) (–7, –8)

68. If a, b, c are in A.P. and if $(b-c)x^2+(c-a)x+(a-b)=0$ and $2(c+a)x^2+(b+c)x=0$ have a common root then

(1) a^2, c^2, b^2 are in A.P. (2) a^2, b^2, c^2 are in A.P.
(3) a^2, c^2, b^2 are in G.P. (4) none of these

69. Let $A = \{x|x^2 + (m - 1)x - 2(m + 1) = 0, x \in R\}$ and $B = \{x \mid (m - 1)x^2 + mx + 1 = 0, x \in R\}$.
Then number of values of m such that $A \cup B$ has exactly 3 distinct elements, is

(1) 4 (2) 5
(3) 6 (4) 7

70. If a quadratic equation $p(x) = 0$ having coefficient of x^2 as unity is such that $p(p(p(x))) = 0$ have a common root, then

(1) $p(0)p(1) > 0$ (2) $p(0)p(1) < 0$
(3) $p(0)p(1) = 0$ (4) $p(0) = 0, p(1) = 0$

Quadratic Function and Location of Roots

71. If x is real, then $\frac{x}{x^2 - 5x + 9}$ lies between

(1) –1 and –1/11 (2) –1/11 and 1
(3) 1 and 1/11 (4) none of these

72. The least integral value of k for which $(k-2)x^2 + 8x + k + 4 > 0$ for all $x \in R$, is

(1) -7 (2) -5
(3) -3 (4) 5

73. The quadratic polynomials defined on real coefficients $P(x) = a_1x^2 + 2b_1x + c_1$ and $Q(x) = a_2x^2 + 2b_2x + c_2$ take positive values $\forall\, x \in R$. What can we say for the trinomial $g(x) = a_1a_2x^2 + b_1b_2x + c_1c_2$?

(1) $g(x)$ takes positive values only
(2) $g(x)$ takes negative values only
(3) $g(x)$ can take positive as well as negative values
(4) nothing definite can be said about $g(x)$

74. If $[x^2 - 2x + a] = 0$ has no solution then, (where [.] represents the greatest integer function)

(1) $-\infty < \alpha < 1$ (2) $2 \le a < \infty$
(3) $1 < \alpha < 2$ (4) $a \in R$

75. The values of a for which $-3 < \dfrac{x^2 + ax - 2}{x^2 + x + 1} < 2$ is valid for all real x is

(1) $(-2, \infty)$ (2) $(-\infty, 1)$
(3) $(-2, 1)$ (4) $(-1, 2)$

76. If the equation $ax^2 + 2bx - 3c = 0$ has non-real roots and $\dfrac{3c}{4} < a + b$ then c is always

(1) < 0 (2) > 0
(3) ≥ 0 (4) zero

77. Let $f(x) = x^2 + ax + b$. If the maximum and the minimum values of $f(x)$ are 3 and 2, respectively for $0 \le x \le 2$, then the possible ordered pair (a, b) is

(1) $(-2, 3)$ (2) $(-3/2, 2)$
(3) $(-5/2, 3)$ (4) $(-5/2, 2)$

78. The value of k for which the equation $3x^2 + 2x(k^2 + 1) + k^2 - 3k + 2 = 0$ has roots of opposite signs, lies in the interval

(1) $(1, 2)$ (2) $(3/2, 2)$
(3) $(-\infty, 0)$ (4) $(-\infty, -1)$

79. If both roots of equation $x^2 - kx + (k+1) = 0$ are positive and greater of the two roots lies in $(2, 3)$, then the true set of real values of k is

(1) ϕ (2) $(-1, 4)$
(3) $(3, 4)$ (4) $(4, \infty)$

80. Let conditions C_1 and C_2 be defined as follows:
$C_1 : b^2 - 4ac \ge 0$, $C_2 : a, -b, c$ are of same sign.
The roots of $ax^2 + bx + c = 0$ are real and positive, if

(1) both C_1 and C_2 are satisfied
(2) only C_2 is satisfied
(3) only C_1 is satisfied
(4) none of these

81. The equation $ax^4 - 2x^2 - (a-1) = 0$ will have real and unequal roots if

(1) $0 < a < 1$ (2) $a > 0, a \ne 1$
(3) $a < 0, a \ne 1$ (4) none of these

82. If exactly one root of the quadratic equation $x^2 - (a+1)x + 2a = 0$ lies in the interval $(0, 3)$ then the set of values a is given by

(1) $(-\infty, 0) \cup (6, \infty)$ (2) $(-\infty, 0]$
(3) $(-\infty, 0] \cup [6, \infty)$ (4) $(0, 6)$

83. If $c < a < b < d$, then roots of the equation $bx^2 + (1 - b(c+d))x + bcd - a = 0$ are

(1) real and one lies between c and a
(2) real and distinct in which one lies between a and b
(3) real and distinct in which one lies between c and d
(4) are not real

84. If the roots of the equation $(m-3)x^2 - 2mx + 5m = 0$ are real and positive then $m \in$

(1) $(2, 15/2]$ (2) $(-3, 15/4]$
(3) $(3, 15/4]$ (4) None of these

85. If the equation $x^2 + 2(k+1)x + 9k - 5 = 0$ has only negative roots, then

(1) $k \ge 6$ (2) $k \le 6$
(3) $k \le 0$ (4) $k \ge 0$

Numerical Value Type

1. If $\sqrt{\sqrt{\sqrt{x}}} = \sqrt[4]{\sqrt[4]{\sqrt[4]{3x^4 + 4}}}$ then the value of x^4 is________.

2. If the equation $x^4 - (3m+2)x^2 + m^2 = 0$ $(m > 0)$ has four real solutions which are in A.P. then the value of 'm' is______.

3. The quadratic polynomial $p(x)$ has the following properties: $p(x) \ge 0$ for all real numbers, $p(1) = 0$ and $p(2) = 2$. The value of $p(11)$ is_______.

4. Let $P(x) = \dfrac{5}{3} - 6x - 9x^2$ and $Q(y) = -4y^2 + 4y + \dfrac{13}{2}$. If there exists a unique pair of real numbers (x, y) such that $P(x)\,Q(y) = 20$, then the value of $|(6x + y)|$ is______.

5. If a and b are positive numbers and each of the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ has real roots, then the smallest possible value of $(a + b)$ is_______.

6. Let α_1, β_1 be the roots of $x^2 - 6x + p = 0$ and α_2, β_2 be the roots of $x^2 - 54x + q = 0$. If $\alpha_1, \beta_1, \alpha_2, \beta_2$ form an increasing G.P., then the value of $(q - p)$ is_______.

7. Given α and β are the roots of the quadratic equation $x^2 - 4x + k = 0$ $(k \ne 0)$. If $\alpha\beta$, $\alpha\beta^2 + \alpha^2\beta$, $\alpha^3 + \beta^3$ are in geometric progression then the value of $7k$ equals_______.

8. If the equation $x^2 + 2(\lambda + 1)x + \lambda^2 + \lambda + 7 = 0$ has only negative roots, then the least value of λ equals_______.

9. If the roots of the cubic equation $x^3 + ax^2 + bx + c = 0$ are three consecutive positive integers, then the value of $\frac{a^2}{5(b+1)}$ is equal to_______.

10. If $a^2 - 4a + 1 = 4$, then the value of $\frac{a^3 - a^2 + a - 1}{a^2 - 1}$ $(a^2 \neq 1)$ is equal to_______.

11. The quadratic equation $x^2 + mx + n = 0$ has roots which are twice those of $x^2 + px + m = 0$ such that $m, n, p \neq 0$. The value of (n/p), is_______.

12. If the coefficient of the quadratic equation $f(x) = 0$ are rational and the coefficient of x^2 is 1. If one root is $\tan \frac{\pi}{8}$ then the value of $f(5)$ is_______.

13. If $ax^2 + bx + 10 = 0$ does not have real and distinct roots, then the maximum value of $b - 5a$ is_______.

14. If the roots of the equation $(x - \alpha)(x - 4 + \beta) + (x - 2 + \alpha)(x + 2 - \beta) = 0$ are p and q then absolute value of the sum of the roots of the equations $2(x - p)(x - q) - (x - \alpha)(x - 4 + \beta) = 0$ and $2(x - p)(x - q) - (x - 2 + \alpha)(x + 2 - \beta) = 0$ is equal to_______.

15. Suppose that α and β are the roots of equation $x^2 - 2x + 5 = 0$. Also, $f(x)$ is a quadratic polynomial whose zeros are $\alpha^3 + \alpha^2 + 20$ and $\beta^3 + 4\beta^2 - 6\beta + 35$. If $f(0) = 120$ then least value of $|f(x)|$ is equal to_______.

16. Let x_1, x_2, x_3 and x_4 be roots of $x^4 + 2x^3 + bx^2 + cx + d = 0$. If $x_1 + x_2 = x_3 + x_4$ and $a, b, c, d \in R$, then $b - c =$_______. (Given that $a = 2$)

17. Let $f_1(x) = ax^2 - bx - c$, $f_2(x) = bx^2 - cx - a$ and $f_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $f_1(\alpha) = f_2(\alpha) = f_3(\alpha)$ then the value of $\frac{2a + 3b + 7c}{a} =$_______.

18. When polynomial $x^{2015} + 7x^3 + 6x^2 + 5x$ is divided by $(x^2 - 1)$, then the quotient is $q(x)$. The coefficient of x in $q(x)$ is_______.

19. If $\sin^2 \alpha$, $\cos^2 \alpha$ and $-\operatorname{cosec}^2 \alpha$ are the zeroes of $P(x) = x^3 + x^2 + ax + b$ $(a, b \in R)$, then $P(2)$ equals _______.

20. Consider the equation $(x^2 - 2x + m)(x^2 - 2x + n) = 0$ (where m and n are real numbers). Let the roots $\alpha, \beta, \gamma, \delta$ $(\alpha < \beta < \gamma < \delta)$ of the equation form an A.P. with first term equal to $\frac{1}{4}$. Then the value of $|m - n|$ is _______.

21. Sum of the values of x satisfying the equation $\sqrt{2x + \sqrt{2x + 4}} = 4$ is _______.

22. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ be c, d. Then the value of $a + b + c + d$, when $a \neq b \neq c \neq d$, is _______.

23. Let ABC be a triangle right angled at C. If $\sin A$ and $\sin B$ are the roots of the quadratic equation $(5n + 8)x^2 - (7n - 20)x + 120 = 0$, $n \in I$, then the value of n is _______.

24. Let α and β be the roots of the equation $x^2 - 5x - 1 = 0$. Then the value of $\frac{\alpha^{15} + \alpha^{11} + \beta^{15} + \beta^{11}}{\alpha^{13} + \beta^{13}}$ is _______.

25. If a is real number, then minimum number of real roots of equation $(x^2 + ax + 1)(3x^2 + ax - 3) = 0$ can be _______.

26. The number of polynomials of the form $x^3 + ax^2 + bx + c$ which are divisible by $x^2 + 1$, where $a, b, c \in \{1, 2, 3, \ldots, 10\}$, must be _______.

27. If α, β and γ are roots of $4x^3 + 8x^2 - x - 2 = 0$, then the value of $\left|\frac{4(\alpha + 1)(\beta + 1)(\gamma + 1)}{\alpha\beta\gamma}\right|$ is _______.

28. Let α and β be the roots of the quadratic equation $px^2 + qx + r = 0$, where $\alpha\beta = 99$. If p, q and r (taken in that order) are in arithmetic progression, then $|(\alpha + \beta)|$ is equal to_______.

29. Value of p $(p > 0)$ for which $(2p)x^2 + (3p + 6)x + (4p + 1)$ is perfect square of a linear expression is _______.

30. $P(x)$ is a quadratic polynomial whose values at $x = 1$ and at $x = 2$ are equal in magnitude but opposite in sign. If -1 is a root of the equation $P(x) = 0$, then the other root is _______.

31. If e^{λ} and $e^{-\lambda}$ are roots of the equation $3x^2 - (a + b)x + 2a = 0$, $a, b, \lambda \in R$, $\lambda \neq 0$, then the least integral value of b is _______.

32. The number of non-negative integral values of k for which the equation $5x^2 + (13 - k)x - 3k - 6 = 0$ has at least one real solution in $(-2, 2)$ is _______.

33. The number of integral value(s) of a so that the graph of $y = 16x^2 + 8(a + 5)x - 7a - 5$ is always above the x-axis is _______.

34. The maximum value of $f(x) = 2bx^2 - x^4 - 3b$ is $g(b)$, where $b > 0$. If b varies, then the value of $|\text{min. } g(b)|$ is _______.

For Questions 35 and 36

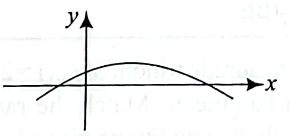
For some real value of k, let $3\alpha^3 - \alpha^2 = k\alpha - 9$, $3\beta^3 - \beta^2 = k\beta - 9$, $3\gamma^3 - \gamma^2 = k\gamma - 9$, where $\alpha > \beta > \gamma$ and $\alpha + \gamma = 0$.

35. The value of k is equal to _______.

36. The value of expression $9(\alpha^{-2} + \beta^{-2} + \gamma^{-2})$ is equal to _______.

Multiple Correct Answers Type

1. If $x^2 - x + a < 0$ for at least one negative x and roots of equation $f(x) = x^2 + (a - 1)x + 4 = 0$ are real and distinct then
   (1) roots of equation $f(x) = 0$ are positive
   (2) one root of $f(x) = 0$ is positive and other is negative
   (3) least integer a is 6
   (4) greatest integer a is –4
2. Let $f(x) = x^3 + x + 1$. If $p(x)$ is a cubic polynomial such that the roots of $p(x) = 0$ are the squares of the roots of $f(x) = 0$, then
   (1) $p(1) = 3$
   (2) the value of $P(n)$, $n \in N$, is odd
   (3) sum of all roots of $p(x) = 0$ is –2
   (4) sum of products of roots taken two at a time is 1
3. If $x, y \in R$ and $2x^2 + 6xy + 5y^2 = 1$, then
   (1) $|x| \le \sqrt{5}$ (2) $|x| \ge \sqrt{5}$
   (3) $y^2 \le 2$ (4) $y^2 \le 4$
4. If the equation $ax^2 + bx + c = 0$, $a, b, c \in R$ have non-real roots then
   (1) $c(a - b + c) > 0$ (2) $c(a + b + c) > 0$
   (3) $c(4a - 2b + c) > 0$ (4) none of these
5. If the equation $ax^2 + bx + c = 0$ $(a > 0)$ has two real roots α and β such that $\alpha < -2$ and $\beta > 2$, then which of following is/are true?
   (1) $a - |b| + c < 0$
   (2) $c < 0$, $b^2 - 4ac > 0$
   (3) $4a - 2|b| + c < 0$
   (4) $9a - 3|b| + c < 0$
6. If the following figure shows the graph of $f(x) = ax^2 + bx + c$, then



   (1) $ac < 0$ (2) $bc > 0$
   (3) $ab > 0$ (4) $abc < 0$
7. If roots of $ax^2 + bx + c = 0$ is α and β and $4a + 2b + c > 0$, $4a - 2b + c > 0$ and $c < 0$, then possible value/vales of $[\alpha] + [\beta]$ is/are (where [.] represents the greatest integer function)
   (1) – 2 (2) – 1
   (3) 0 (4) 1
8. If the quadratic equation $ax^2 + bx + c = 0$ $(a > 0)$ has $\sec^2\theta$ and $\text{cosec}^2\theta$ as its roots then which of the following must hold good?
   (1) $b + c = 0$ (2) $b^2 - 4ac \ge 0$
   (3) $c \ge 4a$ (4) $4a + b \ge 0$
9. Let $a, b, c \in Q^+$ satisfying $a > b > c$. Which of the following statement (s) hold true for the quadratic polynomial $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b)$?
   (1) The graph of the parabola $y = f(x)$ is concave upward.
   (2) Both roots of the equation $f(x) = 0$ are rational.
   (3) x-coordinate of vertex of the graph is positive.
   (4) Product of the roots is always negative.
10. The graph of the quadratic trinomial $y = ax^2 + bx + c$ has its vertex at (4, –5) and two x-intercepts one positive and one negative. Which of the following hold good?
    (1) $a > 0$ (2) $b < 0$
    (3) $c < 0$ (4) $8a = b$
11. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root then it must be equal to
    (1) $\dfrac{pq' - p'q}{q - q'}$ (2) $\dfrac{q - q'}{p' - p}$
    (3) $\dfrac{p' - p}{q - q'}$ (4) $\dfrac{pq' - p'q}{p - p'}$
12. If $x^3 + 3x^2 - 9x + c$ is of the form $(x - \alpha)^2(x - \beta)$ then c is equal to
    (1) 27 (2) – 27
    (3) 5 (4) – 5
13. If $\dfrac{x^2 + ax + 3}{x^2 + x + a}$, takes all real values for possible real values of x, then
    (1) $4a^3 + 39 < 0$ (2) $4a^3 + 39 \ge 0$
    (3) $a \ge 1/4$ (4) $a < 1/4$
14. If the roots of the equation $x^2 + ax + b = 0$ are c and d, then roots of the equation $x^2 + (2c + a)x + c^2 + ac + b = 0$ are
    (1) c (2) $d - c$
    (3) $2c$ (4) 0
15. For the quadratic equation $x^2 + 2(a + 1)x + 9a - 5 = 0$ which of the following are true?
    (1) If $2 < a < 5$ then roots are of opposite signs.
    (2) If $a < 0$, then roots are of opposite signs.
    (3) If $a > 7$, then both roots are negative.
    (4) If $2 \le a \le 5$ then roots are unreal.

Linked Comprehension Type

For Questions 1 to 3

Consider $f(x) = x^2 + b_1x + c_1$ and $g(x) = x^2 + b_2x + c_2$ such that real roots of $f(x) = 0$ are α, β and real roots of $g(x) = 0$ are $\alpha + h$, $\beta + h$. Also, the least value of $f(x)$ is –1/4 and the least value of $g(x)$ occurs at $x = 7/2$.

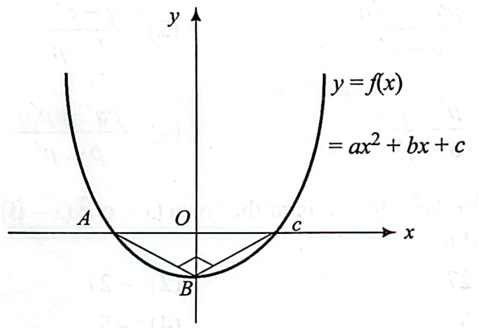
1. The least value of $g(x)$ is
(1) $-1/4$ (2) -1
(3) $-1/3$ (4) $-1/2$

2. The value of b_2 is
(1) -5 (2) 9
(3) -8 (4) -7

3. The roots of $f(x) = 0$ are
(1) 3, -4 (2) -3, 4
(3) 3, 4 (4) -3, -4

For Questions 4 to 6

In the given figure, vertices of ΔABC lie on $y = f(x) = ax^2 + bx + c$. ΔABC is right angled isosceles triangle whose hypotenuse AC measures $4\sqrt{2}$ units.



4. $y = f(x)$ is given by
(1) $y = x^2 - 2\sqrt{2}$ (2) $y = x^2 - 12$
(3) $y = \frac{x^2}{2} - 2$ (4) $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

5. Minimum value of $y = f(x)$ is
(1) -4 (2) -2
(3) $-2\sqrt{2}$ (4) none of these

6. Number of integral values of k for which one root of $f(x) = 0$ is more than k and other less than k is
(1) 6 (2) 4
(3) 5 (4) 7

For Questions 7 and 8

Let $f(x) = ax^2 + bx + c$, $a \neq 0$, $a, b, c \in I$. Suppose that $f(1) = 0$, $50 < f(7) < 60$ and $70 < f(8) < 80$.

7. The least value of $f(x)$ is
(1) 3/4 (2) 9/2
(3) $-9/8$ (4) $-3/4$

8. If $f(x) < 0$, then number of integral values of x is
(1) 0 (2) 1
(3) 2 (4) 3

For Questions 9 to 11

Let $x^2 - ax + b = 0$ and $x^2 - px + q = 0$ be two quadratic equations.

9. If the given equations have one common root and the other roots are reciprocals of each other, then $(q - b)^2$ equals
(1) $bq(p - a)^2$ (2) $b(p - a)^2$
(3) $q(p - a)^2$ (4) none of these

10. If the two equations have one root is common and the second equation has equal roots, then $b + q$ is equal to
(1) $-ap$ (2) ap
(3) $(ap)/2$ (4) $2ap$

11. Let α, β be the roots of first equation and γ, δ be the roots of second equation. If $\alpha, \beta, \gamma, \delta$ are in A.P., then $q - b$ equals
(1) $p^2 - a^2$ (2) $a^2 - p^2$
(3) $(p^2 - a^2)/4$ (4) none of these

For Questions 12 to 14

The roots of the equation $x^2 - ax + a + 5 = 0$ are real and each is one more than the respective roots of the equation, $x^2 + px + q = 0$. Also, minimum positive integral value of p is k, where $a, p, q \in R$.

12. The value of q lies in the interval
(1) (0, 2) (2) (3, 5)
(3) (5, 7) (4) (7, 9)

13. If $x^2 - ax + a + 5 < k$ for at least one real x, then 'a' cannot be
(1) -2 (2) 2
(3) 5 (4) 10

14. Minimum value of $(a^2 - 4a)$ is equal to
(1) 5 (2) 10
(3) 15 (4) 20

Matrix Match Type

1. Consider the quadratic trinomial $f(x) = 2x^2 - 10px + 7p - 1$, where p is a parameter. Match the range of p given in Column II with the conditions given in Column I.

	Column I		Column II
I	Both roots of $f(x) = 0$ are confined in $(-1, 1)$.	(p)	$(2/5, \infty)$
II	Exactly one root of $f(x) = 0$ lies in $(-1, 1)$.	(q)	ϕ
III	Both roots of $f(x) = 0$ are greater than 1.	(r)	$(-1/17, 1/3)$
IV	One root of $f(x) = 0$ is greater than 1 and other root of $f(x) = 0$ is less than -1.	(s)	$(-\infty, -1/17] \cup [1/3, \infty)$
		(t)	$(-\infty, -1/17) \cup (1/3, \infty)$

2. Match the following columns:

	Column I		Column II
I	If a, b, c and d are four non-zero real numbers such that $(d+a-b)^2 + (d+b-c)^2 = 0$ and the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are real and equal then	(p)	$a+b+c=0$
II	If a, b and c are non-zero positive real numbers such that $\log a$, $\log b$ and $\log c$ are in A.P. then	(q)	a, b, c are in A.P.
III	If the equation $ax^2+bx+c=0$ and $x^3-3x^2+3x-1=0$ have a common real root then	(r)	a, b, c are in G.P.
IV	If a, b and c are positive real numbers such that the expression $bx^2 + \left(\sqrt{(a+c)^2+4b^2}\right)x + (a+c)$ is non negative $\forall\, x \in R$ then	(s)	a, b, c are in H.P.

3. Match the following columns for the equation $x^2 + a\,\lvert x\rvert + 1 = 0$ where 'a' is a parameter.

	Column I		Column II
I	No real roots	(p)	$a < -2$
II	Two real roots	(q)	ϕ
III	Three real roots	(r)	$a = -2$
IV	Four distinct real roots	(s)	$a \geq 0$

4. Column I contains rational algebraic expression and column II contains possible integers which lie in their ranges. Match the columns.

	Column I		Column II
I	$y = \dfrac{x^2-2x+4}{x^2+2x+4}, x \in R$	(p)	1
II	$y = \dfrac{x^2-3x-2}{2x-3}, x \in R$	(q)	4
III	$y = \dfrac{2x^2-2x+4}{x^2-4x+3}, x \in R$	(r)	−3
IV	$x^2-(a-3)x+2<0, \forall\, x \in (-2, 3)$	(s)	−10

JEE ARCHIVES

JEE (Main) 2024 Questions

Single Correct Answer Type

1. If α, β are the roots of the equation, $x^2 - x - 1 = 0$ and $S_n = 2023\alpha^n + 2024\beta^n$, then
(1) $2S_{12} = S_{11} + S_{10}$ (2) $S_{12} = S_{11} + S_{10}$
(3) $2S_{11} = S_{12} + S_{10}$ (4) $S_{11} = S_{10} + S_{12}$

[Nature of Roots of Quadratic Equations]

2. Let S be the set of positive integral values of a for which $\frac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32} < 0, \forall x \in R$. Then, the number of elements in S is
(1) ∞ (2) 3
(3) 0 (4) 1

[Graph of Quadratic Function and Inequalities]

3. The number of solutions of the equation $e^{\sin x} - 2e^{-\sin x} = 2$, is
(1) 0 (2) more than 2
(3) 2 (4) 1

[Equations Reducible to Quadratic]

4. Let $S = \{x \in R : (\sqrt{3}+\sqrt{2})^x + (\sqrt{3}-\sqrt{2})^x = 10\}$. Then the number of elements in S is
(1) 4 (2) 0
(3) 2 (4) 1

[Equations Reducible to Quadratic]

5. Let the sum of the maximum and the minimum values of the function $f(x) = \frac{2x - 3x + 8}{2x + 3x + 8}$ be $\frac{m}{n}$, where gcd(m, n) = 1. Then $m + n$ is equal to
(1) 217 (2) 195
(3) 182 (4) 201

[Graph of Quadratic Function and Inequalities, Locations of Roots of Quadratic Equation]

6. If 2 and 6 are the roots of the equation $ax^2 + bx + 1 = 0$, then the quadratic equation, whose roots are $\frac{1}{2a+b}$ and $\frac{1}{6a+b}$ is
(1) $x^2 + 8x + 12 = 0$ (2) $4x^2 + 14x + 12 = 0$
(3) $2x^2 + 11x + 12 = 0$ (4) $x^2 + 10x + 16 = 0$

[Quadratic Equations, Equations Reducible to Quadratic, Nature of Roots of Quadratic Equations]

7. Let α, β be the distinct roots of the equation $x^2 - (t^2 - 5t + 6)x + 1 = 0$, $t \in R$ and $a_n = \alpha^n + \beta^n$.
Then the minimum value of $\frac{a_{2023} + a_{2025}}{a_{2024}}$ is
(1) $-1/2$ (2) $1/2$
(3) $1/4$ (4) $-1/4$

[Quadratic Equations, Equations Reducible to Quadratic, Nature of Roots of Quadratic Equations]

8. Let α, β be the roots of the equation $x^2 + 2\sqrt{2}x - 1 = 0$. The quadratic equation, whose roots are $\alpha^4 + \beta^4$ and $\frac{1}{10}(\alpha^6 + \beta^6)$, is
(1) $x^2 - 195x + 9506 = 0$ (2) $x^2 - 180x + 9506 = 0$
(3) $x^2 - 190x + 9466 = 0$ (4) $x^2 - 195x + 9466 = 0$

[Quadratic Equations, Equations Reducible to Quadratic, Nature of Roots of Quadratic Equations]

9. Let α, β; $\alpha > \beta$, be the roots of the equation $x^2 - \sqrt{2}x - \sqrt{3} = 0$. Let $P_n = \alpha^n - \beta^n, n \in N$. Then $(11\sqrt{3} - 10\sqrt{2})P_{10} + (11\sqrt{2} + 10)P_{11} - 11P_{12}$ is equal to
(1) $11\sqrt{2}P_9$ (2) $10\sqrt{2}P_9$
(3) $10\sqrt{3}P_9$ (4) $11\sqrt{3}P_9$

[Quadratic Equations, Equations Reducible to Quadratic, Nature of Roots of Quadratic Equations]

Numerical Value Type

1. Let α, β be the roots of the equation $x^2 - x + 2 = 0$ with $\text{Im}(\alpha) > \text{Im}(\beta)$. Then $\alpha^6 + \alpha^4 + \beta^4 - 5\alpha^2$ is equal to ______.

[Nature of Roots of Quadratic Equations]

2. Let the set $C = \{(x, y) | x^2 - 2^y = 2023, x, y \in N\}$. Then $\sum_{(x,y)\in C}(x+y)$ is equal to __________.

[Equations Reducible to Quadratic]

3. Let $\alpha, \beta \in N$ be roots of the equation $x^2 - 70x + \lambda = 0$, where $\frac{\lambda}{2}, \frac{\lambda}{3} \notin N$. If λ assumes the minimum possible value, then $\frac{(\sqrt{\alpha - 1} + \sqrt{\beta - 1})(\lambda + 35)}{|\alpha - \beta|}$ is equal to ______.

[Nature of Roots of Quadratic Equations]

4. Let a, b, c be the lengths of three sides of a triangle satisfying the condition $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$. If the set of all possible values of x is the interval (α, β), then $12(\alpha^2 + \beta^2)$ is equal to __________.

[Nature of Roots of Quadratic Equations]

5. Let α, β be roots of $x^2 + \sqrt{2}x - 8 = 0$. If $U_n = \alpha^n + \beta^n$, then $\frac{U_{10} + \sqrt{2}U_9}{2U_8}$ is equal to _______.

[Quadratic Equations, Equations Reducible to Quadratic, Nature of Roots of Quadratic Equations]

6. The number of distinct real roots of the equation $|x + 1||x + 3| - 4|x + 2| + 5 = 0$, is ________.

[Quadratic Equations, Equations Reducible to Quadratic, Nature of Roots of Quadratic Equations]

JEE (Advanced) Questions (2010-2024)

Single Correct Answer Type

1. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$, and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha+\beta=-p$ and $\alpha^3+\beta^3=q$, then a quadratic equation having α/β and β/α as its roots is
   (1) $(p^3+q)x^2-(p^3+2q)x+(p^3+q)=0$
   (2) $(p^3+q)x^2-(p^3-2q)x+(p^3+q)=0$
   (3) $(p^3-q)x^2-(5p^3-2q)x+(p^3-q)=0$
   (4) $(p^3-q)x^2-(5p^3+2q)x+(p^3-q)=0$
   (IIT-JEE, 2010)

2. A value of b for which the equations $x^2+bx-1=0$, $x^2+x+b=0$ have one root in common is
   (1) $-\sqrt{2}$ (2) $-i\sqrt{3}$
   (3) $\sqrt{2}$ (4) $\sqrt{3}$ **(IIT-JEE, 2011)**

3. Let α and β be the roots of $x^2-6x-2=0$, with $\alpha>\beta$. If $a_n=\alpha^n-\beta^n$ for $n \geq 1$, then the value of $\dfrac{a_{10}-2a_8}{2a_9}$ is
   (1) 1 (2) 2
   (3) 3 (4) 4 **(IIT-JEE, 2011)**

4. The quadratic equation $p(x)=0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x))=0$ has
   (1) only purely imaginary roots
   (2) all real roots
   (3) two real and two purely imaginary roots
   (4) neither real nor purely imaginary roots
   (JEE Advanced 2014)

5. Let $-\dfrac{\pi}{6}<\theta<-\dfrac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2-2x\sec\theta+1=0$ and α_2 and β_2 are the roots of the equation $x^2+2x\tan\theta-1=0$. If $\alpha_1>\beta_1$ and $\alpha_2>\beta_2$, then $\alpha_1+\beta_2$ equals
   (1) $2(\sec\theta-\tan\theta)$ (2) $2\sec\theta$
   (3) $-2\tan\theta$ (4) 0
   (JEE Advanced 2016)

6. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2+20x-2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2-20x+2020$. Then the value of $ac(a-c)+ad(a-d)+bc(b-c)+bd(b-d)$ is
   (1) 0 (2) 8000
   (3) 8080 (4) 16000
   (JEE Advanced 2020)

Numerical Value Type

1. The number of distinct real roots of $x^4-4x^3+12x^2+x-1=0$ is ________. **(IIT-JEE, 2011)**

2. For $x \in R$, the number of real roots of the equation $3x^2-4|x^2-1|+x-1=0$ is ________.
   (JEE Advanced 2021)

3. Let $f(x)=x^4+ax^3+bx^2+c$ be a polynomial with real coefficients such that $f(1)=-9$. Suppose that $i\sqrt{3}$ is a root of the equation $4x^3+3ax^2+2bx=0$, where $i=\sqrt{-1}$. If α_1, α_2, α_3 and α_4 are all the roots of the equation $f(x)=0$, then $|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2$ is equal to ________.
   (JEE Advanced 2024)

Multiple Correct Answers Type

1. Let S be the set of all non-zero real numbers such that the quadratic equation $\alpha x^2-x+\alpha=0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1-x_2|<1$. Which of the following intervals is(are) a subset(s) of S?
   (1) $\left(-\dfrac{1}{2},-\dfrac{1}{\sqrt{5}}\right)$ (2) $\left(-\dfrac{1}{\sqrt{5}},0\right)$
   (3) $\left(0,\dfrac{1}{\sqrt{5}}\right)$ (4) $\left(\dfrac{1}{\sqrt{5}},\dfrac{1}{2}\right)$
   (JEE Advanced 2015)

2. Let $S=\{(a, b, c): a, b, c \in R$ and $ax^2+2bxy+cy^2>0$ for all $(x, y) \in R^2-\{(0, 0)\}\}$.
   Then which of the following statements is (are) TRUE?
   (1) $\left(2, \dfrac{7}{2}, 6\right) \in S$
   (2) If $\left(3, b, \dfrac{1}{12}\right) \in S$ then $|2b|<1$
   (3) For any given $(a, b, c) \in S$, the system of linear equations
   $$ax+by=1$$
   $$bx+cy=-1$$
   has a unique solution.
   (4) For any given $(a, b, c) \in S$, the system of linear equations
   $$(a+1)x+by=0$$
   $$bx+(c+1)y=0$$
   has a unique solution. **(JEE Advanced 2024)**

Linked Comprehension Type

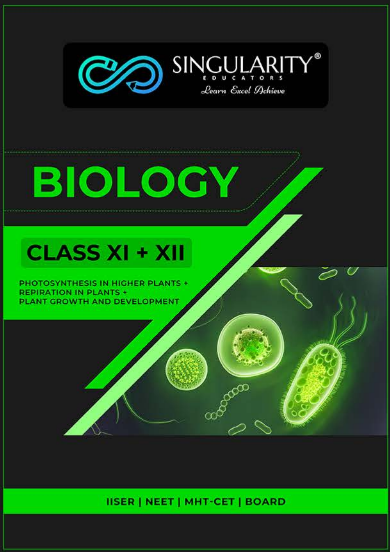
For Problems 1 and 2

Let p, q be integers and let α, β be the roots of the equation, $x^2-x-1=0$ where $\alpha \neq \beta$. For $n=0, 1, 2, \ldots$, let $a_n=p\alpha^n+q\beta^n$.

Fact: If a and b are rational numbers and $a+b\sqrt{5}=0$, then $a=0=b$.

1. $a_{12}=$
   (1) $a_{11}-a_{10}$ (2) $a_{11}+a_{10}$
   (3) $2a_{11}+a_{10}$ (4) $a_{11}+2a_{10}$

2. If $a_4=28$, then $p+2q=$
   (1) 21 (2) 14
   (3) 7 (4) 12
   (JEE Advanced 2017)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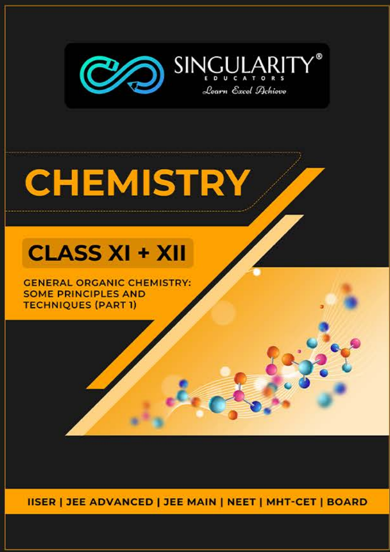
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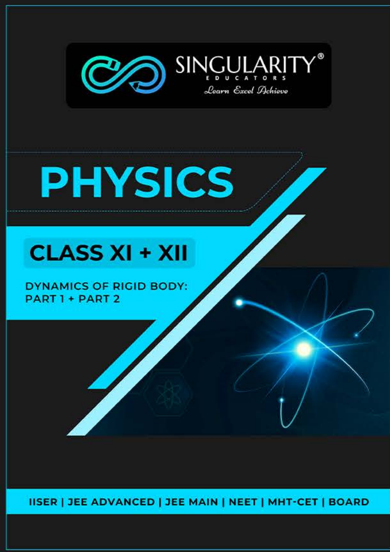
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