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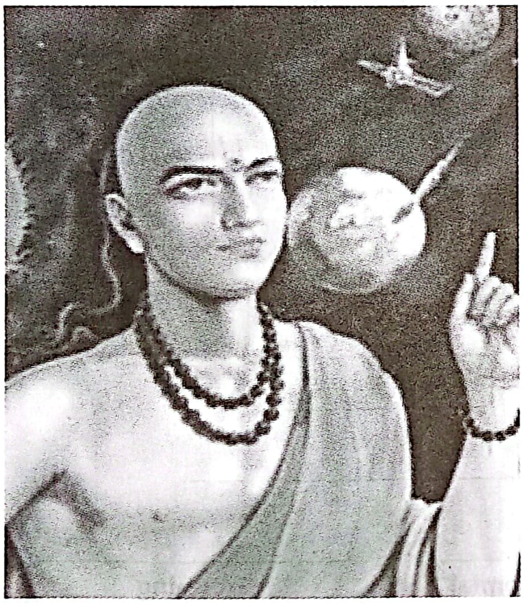
MATHS

CLASS IX
FOUNDATION

3 ALGEBRA

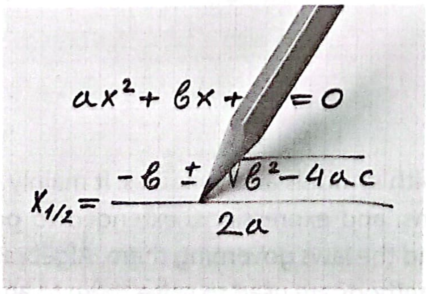
Chapter Outline



Bhāskara
(1114 AD–1185 AD)

Bhāskara, also known as Bhāskarācārya and as Bhāskara II, was an Indian mathematician and astronomer. Born in a Deshastha Brahmin family of scholars, mathematicians, and astronomers, Bhaskara was the leader of a cosmic observatory at Ujjain, the main mathematical center of ancient India. Bhāskara and his works represent a significant contribution to mathematical and astronomical knowledge in the 12th century.

He has been called the greatest mathematician of medieval India. His main work Siddhānta-Śiromani, (Sanskrit for "Crown of Treatises") is divided into four parts called Līlāvatī, Bījagaṇita, Grahagaṇita and Golādhyāya, which are also sometimes considered four independent works. These four sections deal with arithmetic, algebra, mathematics of the planets, and spheres, respectively.



3.01 Polynomials

MIND MAP

ALGEBRAIC EXPRESSION

- **Special Products and Identities**
  - $(x+a)(x+b)=x^2+(a+b)x+ab$
  - $(a+b)^2=a^2+2ab+b^2$
  - $(a-b)^2=a^2-2ab+b^2$
  - $(a+b)(a-b)=a^2-b^2$
  - $(a+b)^3=a^3+3a^2b+3ab^2+b^3$
  - $(a-b)^3=a^3-3a^2b+3ab^2-b^3$
  - $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$
- **Polynomials**
  - **Types**
    - Linear
    - Quadratic
    - Cubic
    - Constant
  - **Factorization**
    - **Factor out the GCF**
      $2yx^2-8xy-24y$
      $=2y(x^2-4x-12)$
    - **Special Cases**
      - **Difference of Two Squares**
        $x^2-9=x^2-3^2$
        $=(x+3)(x-3)$
      - **Perfect Square Trinomial**
        $x^2-10x+25=(x-5)^2$
      - **Grouping**
        $4x^2-4x-15$
        $=4x^2-10x+6x-15$
        $=2x(2x-5)+3(2x-5)$
        $=(2x-5)(2x+3)$
- **Exponents**
  - $\sqrt[n]{a}=a^{\frac{1}{n}}$
  - $a^p.a^q=a^{p+q}$
  - $\frac{a^p}{a^q}=a^{p-q}$
  - $(a^p)^q=a^{p\times q}$
  - $a^pb^p=(ab)^p$

INTRODUCTION

Algebra is a branch of mathematics that deals with symbols and variables. It mainly consists of arithmetic operations performed in the form of statements, equations, and expressions, extended to polynomials using certain special products and identities along with exponents and the laws governing them. Algebraic equations have different forms of representation, and consecutively, different methods are used to solve them to analyze various complex situations. These algebraic equations can be dealt with mathematically as well as graphically.

ALGEBRAIC IDENTITY

An algebraic identity is an equality or, in simpler terms, an algebraic equation, which is true for all values of the variables it contains.

Examples

(i) In the equation $x + 12 = 15$, the mathematical statement holds true for only a particular value of x, that is, for $x = 3$. Hence, it is not an identity.

(ii) In the equation $5x - 4x = x$, it is clear that whatever is the value of x, the statement always hold true. Hence, it becomes an identity.

Some Special Products and Identities

I. $(x+a)(x+b) = x^2 + (a+b)x + ab$

Proof: $(x+a)(x+b) = x(x+b) + a(x+b)$

$$= x^2 + bx + ax + ab$$
$$= x^2 + ax + bx + ab$$
$$= x^2 + (a+b)x + ab$$

II. $(a+b)^2 = a^2 + 2ab + b^2$

Proof: $(a+b)^2 = (a+b)(a+b)$

$$= a(a+b) + b(a+b)$$
$$= a^2 + ab + ba + b^2$$
$$= a^2 + 2ab + b^2$$

III. $(a-b)^2 = a^2 - 2ab + b^2$

Proof: $(a-b)^2 = (a-b)(a-b)$

$$= a(a-b) - b(a-b)$$
$$= a^2 - ab - ba + b^2$$
$$= a^2 - 2ab + b^2$$

IV. $(a+b)(a-b) = a^2 - b^2$

Proof: $(a+b)(a-b) = a(a-b) + b(a-b)$

$$= a^2 - ab + ba - b^2$$
$$= a^2 - b^2$$

V. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

Proof: $(a+b+c)^2 = (a+b+c)(a+b+c)$

$$= a(a+b+c) + b(a+b+c) + c(a+b+c)$$
$$= a^2 + ab + ac + ba + b^2 + bc + ca + cb + c^2$$
$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

VI. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Proof: $(a+b)^3 = (a+b)^2(a+b)$

$$= \left(a^2 + 2ab + b^2\right)(a+b)$$
$$= a\left(a^2 + 2ab + b^2\right) + b\left(a^2 + 2ab + b^2\right)$$

$$= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

VII. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Proof: $(a-b)^3 = (a-b)^2(a-b)$
$$= \left(a^2 - 2ab + b^2\right)(a-b)$$
$$= a\left(a^2 - 2ab + b^2\right) - b\left(a^2 - 2ab + b^2\right)$$
$$= a^3 - 2a^2b + ab^2 - ba^2 + 2ab^2 - b^3$$
$$= a^3 - 3a^2b + 3ab^2 - b^3$$

Key points

- $(a+b)^2 + (a-b)^2 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2$
  $= 2a^2 + 2b^2$
  $= 2\left(a^2 + b^2\right)$
- $(a+b)^2 - (a-b)^2 = a^2 + 2ab + b^2 - a^2 + 2ab - b^2$
  $= 4ab$

Exponents

An exponent is a real number that expresses the power to which a quantity must be raised or lowered. For instance, in 2^4, 2 is called the base which is raised to the exponent 4. This means that 2 is to be multiplied by itself 4 times, that is, $2^4 = 2 \times 2 \times 2 \times 2$. Similarly, $3^5 = 3 \times 3 \times 3 \times 3 \times 3$.

In general, if x is any real number and n is a natural number, then

$$x^n = \underbrace{x \cdot x \cdot x \ldots x \cdot x}_{n \text{ times}}$$

The form $x \cdot x \cdot x \ldots x \cdot x$ is called the **product form**, whereas the form x^n is called the **exponential form**. The process of writing a number in exponential form is called **exponentiation**.

Examples

(i) We can write $-5 \times -5 \times -5 \times -5$ in the exponential form as $(-5)^4$ and it is read as "–5 raised to the power 4." Here, –5 is the base and 4 is the exponent.

(ii) We can write $\left(-\frac{4}{3}\right)^2$ in the product form as $-\frac{4}{3} \times -\frac{4}{3}$ which is equal to $\frac{16}{9}$.

Key point

$$(-x)^n = \begin{cases} x^n, \text{ if } n \text{ is even} \\ -x^n, \text{ if } n \text{ is odd} \end{cases}$$

Rules of Exponents

Operations on exponents are governed by rules that are listed in the following table:

Rule Name	Definition	Example
Product Rule	$a^n \cdot a^m = a^{n+m}$	$2^3 \cdot 2^4 = 2^{3+4} = 2^7 = 128$
	$a^n \cdot b^n = (a \cdot b)^n$	$3^2 \cdot 4^2 = (3 \cdot 4)^2 = 12^2 = 144$
Quotient Rule	$\frac{a^n}{a^m} = a^{n-m}$	$\frac{2^5}{2^3} = 2^{5-3} = 2^2 = 4$
	$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$	$\frac{4^3}{2^3} = \left(\frac{4}{2}\right)^3 = 2^3 = 8$
Power Rule	$(a^n)^m = a^{nm}$	$(2^3)^2 = 2^{3\times 2} = 2^6 = 64$
	$a^{n^m} = a^{(n^m)}$	$2^{3^2} = 2^{(3^2)} = 2^9 = 512$
	$\sqrt[m]{(a^n)} = a^{\frac{n}{m}}$	$\sqrt[2]{(2^6)} = 2^{\frac{6}{2}} = 2^3 = 8$
	$a^{\frac{1}{n}} = \sqrt[n]{a}$	$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$
Negative Exponent Rule	$a^{-n} = \frac{1}{a^n}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8} = 0.125$
Zero Rule	$a^0 = 1$	$5^0 = 1$
	$0^n = 0$ for $n > 0$	$0^5 = 0$
One Rule	$a^1 = a$	$5^1 = 5$
	$1^n = 1$	$1^5 = 1$
Minus One Rule	$(-1)^n = \begin{cases} 1 & ,n \text{ even} \\ -1 & ,n \text{ odd} \end{cases}$	$(-1)^5 = -1$

Key points

- The form 0^0 is undefined, unless otherwise stated.
- The reciprocal of a non-zero integer a is $\frac{1}{a}$ and it is denoted by a^{-1}, that is, $a^{-1} = \frac{1}{a}$. For a fractional number $\frac{p}{q}$, where $p \neq 0, q \neq 0$, we have $\left(\frac{p}{q}\right)^{-1} = \frac{q}{p}$.

Examples

(i) $(x+2)^3 \cdot (x+2)^5 = (x+2)^{3+5} = (x+2)^8$

(ii) $x^2 \cdot x^6 = x^{2+6} = x^8$

(iii) $\dfrac{x^7}{x^5} = x^{7-5} = x^2$

(iv) $\left[\left(\dfrac{2}{3}\right)^2\right]^3 = \left(\dfrac{2}{3}\right)^{2\times3} = \left(\dfrac{2}{3}\right)^6 = \dfrac{2^6}{3^6} = \dfrac{2\times2\times2\times2\times2\times2}{3\times3\times3\times3\times3\times3} = \dfrac{64}{729}$

(v) $\left[\left(\dfrac{-1}{3}\right)^5\right]^2 = \left(\dfrac{-1}{3}\right)^{5\times2} = \left(\dfrac{-1}{3}\right)^{10} = \dfrac{(-1)^{10}}{3^{10}} = \dfrac{1}{59049}$

(vi) $\left(\dfrac{3}{5}\right)^{-2} = \left(\dfrac{5}{3}\right)^2 = \dfrac{25}{9}$

(vii) $\left(\dfrac{-3}{8}\right)^{-3} = \left(\dfrac{8}{-3}\right)^3 = \dfrac{8^3}{(-3)^3} = \dfrac{8\times8\times8}{(-3)\times(-3)\times(-3)} = -\dfrac{512}{27}$

(viii) $\dfrac{27a^3b^6c^2}{3a^2bc} = 9a^{3-2}b^{6-1}c^{2-1} = 9ab^5c$

MISCONCEPTION

$a^3 \cdot b^3 = (ab)^{3+3} = (ab)^6$

FACT: $a^3 \cdot b^3 = (ab)^3 \neq (ab)^6$

POLYNOMIALS

An algebraic expression consisting of two or more algebraic terms is called a polynomial. The word "polynomial" is derived from the Greek word *poly*, meaning "many," and the Latin word *nomen*, meaning "name" or "term", in this case.

Degree of a polynomial in one variable is defined as the highest power of the variable in the polynomial. If a polynomial contains two or more variables, then the sum of the powers of all the variables in each term is calculated and the highest sum thus obtained becomes the degree of the polynomial. Example, $6x^3 - 2x^2y + xy^2 - 3x^2y^2$ is a polynomial in x and y of degree 4.

Key point

Variables in a polynomial have only whole number powers.

Types of Polynomials

1. **Linear:** A polynomial of degree 1 is called a linear polynomial. For example, $3+5x$, $6-2y$, and $\dfrac{3}{2}+7z$ are linear polynomials.

2 **Quadratic:** A polynomial of degree 2 is called a quadratic polynomial. For example, $6x^2 - 5x + 4$ and $2y^2 + 3y - 1$ are quadratic polynomials.

3. **Cubic:** A polynomial of degree 3 is called a cubic polynomial. For example, x^3+5 and x^3+2x^2+4x-3 are cubic polynomials.
4. **Constant:** A polynomial having only constant terms is called a constant polynomial. Its degree is 0. For example, 6 is a constant polynomial.

MISCONCEPTION

$ax^2+b\frac{1}{x}+c$ is a polynomial.

FACT: ax^2+bx+c is a polynomial but $ax^2+b\frac{1}{x}+c$ is not a polynomial as it can be rewritten as $ax^2+bx^{-1}+c$ and power of a polynomial term can never be negative.

ACTIVITY

Aim: To identify the degree and the type of the polynomial.

Illustration: The highest power of the variable in a polynomial of one variable is called the degree of the polynomial. The largest sum of the powers of all variables in each term in a polynomial of two or more variables is called the degree of the polynomial.

Polynomial	Degree	Name of the Polynomial
$\frac{5}{2}-p$	1	Linear polynomial
$3x^2+5x-7$	2	Quadratic polynomial
x^3+2x^2+4x-3	3	Cubic polynomial
32	0	Constant polynomial

LCM of Polynomials

The LCM of two or more polynomials is the lowest degree polynomial that is exactly divisible by each of the given polynomials.

Example: Find the LCM of $24ab$, $4a^2b$, $26a^2b^2$.

Solution: LCM of $24ab$, $4a^2b$, $26a^2b^2$ = LCM (24, 4, 26) × LCM (ab, a^2b, a^2b^2) = $312a^2b^2$

HCF of Polynomials

The HCF of two or more polynomials is the highest degree polynomial that exactly divides each of the given polynomials.

Example: Find the HCF of $32ab$, $4a^2b$, $28a^2b^2$.

Solution: HCF of $32ab$, $4a^2b$, $28a^2b^2$ = HCF (32, 4, 28) × HCF $(ab, a^2b, a^2b^2) = 4ab$

Factor Form and Expanded Form of a Polynomial

Consider the polynomial $(x+2)(x+4)$. It can be rewritten as follows:

$$(x+2)(x+4) = x(x+4)+2(x+4)$$
$$= x^2+4x+2x+8$$
$$= x^2+6x+8$$

The polynomial x^2+6x+8 is called the expanded form of the factor form $(x+2)(x+4)$.

Example: Which of the following polynomials are in factor form?

(i) $(5x-7)$ **(ii)** $(3x+5)(6x-7)$ **(iii)** $(8x+6)(3x+2)-7$

(iv) $9x(7y-z)$ **(v)** $3y(x+z)-5$ **(vi)** $4x^2+y^2-z+6$

Solution: Only (i), (ii), and (iv) are in factor form.

Key point

The factor form of a polynomial is always a single term which is the product of its factors, whereas the expanded form of a polynomial has at least two terms, each of which is a monomial.

For instance, the factor form $5x(x+7)$ is a single term which is the product of 5, x, and $(x+7)$. The expanded form $5x^2+35x$ has two terms in it, each of which is a monomial.

Factorization of Polynomials

Now that we know how to write the expanded form of a polynomial given in its factor form, we should also be able to do the vice versa. The process of converting a polynomial from its expanded form into factor form is called factorization of the polynomial.

Factorization can be accomplished by many methods. Let us discuss some of these in detail.

1. Factorization by Common Factors

Case I: Each term of the given polynomial contains a common monomial factor.

Here, write each term in prime factor notation, take out the factor common to both the terms, and put the remaining terms in a bracket.

Example: Factorize $6ab^3+9a^2b^2$.

Solution: $6ab^3+9a^2b^2$ can be written in the prime factor notation as follows:

$$3\times 2a\times b\times b\times b+3\times 3\times a\times a\times b\times b = 3\times a\times b\times b(2b+3a) = 3ab^2(2b+3a)$$

Case II: Each term of the given polynomial is a multiple of another polynomial.

Here, take out the common polynomial from each term, put the remaining terms in a bracket, and follow the procedure in case I for the terms in the bracket.

Example: Factorize $2a(a^2+1)-6(a^2+1)$.

Solution: Since a^2+1 is common in both the terms, we take it out and proceed as shown below:

$$2a(a^2+1)-6(a^2+1)=(a^2+1)(2a-6)$$
$$=(a^2+1)2\,(a-3)$$
$$=2(a^2+1)(a-3)$$

2. Factorization by Grouping

In this method, terms of the given polynomial are written in groups such that each group has a common factor and the remainder expression is same in all the groups. Further simplification is carried out by distributive laws.

Example: Factorize $2a^3-3b^3+2ab^2-3a^2b$.

Solution: $2a^3-3b^3+2ab^2-3a^2b=\left(2a^3+2ab^2\right)-\left(3a^2b+3b^3\right)$

$$=2a\left(a^2+b^2\right)-3b\left(a^2+b^2\right)$$
$$=\left(a^2+b^2\right)(2a-3b)$$

3. Factorization by Using Identities

Some of the common identities used to factorize polynomials are listed below:

(i) $a^2+b^2+2ab=(a+b)^2$

(ii) $a^2+b^2-2ab=(a-b)^2$

(iii) $a^2-b^2=(a+b)(a-b)$

(iv) $(a+b)^3=a^3+b^3+3ab(a+b)$

(v) $a^3+b^3=(a+b)(a^2+b^2-ab)$

(vi) $(a-b)^3=a^3-b^3-3ab(a-b)$

(vii) $a^3-b^3=(a-b)(a^2+b^2+ab)$

Example: Factorize $p^2-10p+25$.

Solution: $p^2-10p+25=(p)^2-2\times5\times p+(5)^2$

$$=(p-5)^2 \qquad \left[\because (a-b)^2=a^2+b^2-2ab\right]$$

Example: Factorize $32a^3+108b^3$.

Solution: $32a^3+108b^3=4(8a^3+27b^3)=4\left[(2a)^3+(3b)^3\right]$

Since $a^3+b^3=(a+b)(a^2+b^2-ab)$, the factorization becomes

$$32a^3+108b^3=4\left[(2a+3b)\left((2a)^2+(3b)^2-(2a)(3b)\right)\right]$$
$$=4(2a+3b)(4a^2+9b^2-6ab)$$

4. Factorization of Trinomials of the form x^2+px+q

Here we determine two numbers a and b such that $a+b=p$ and $ab=q$. Then

$$x^2+px+q=x^2+(a+b)x+(ab)$$
$$=x^2+ax+bx+ab$$
$$=x(x+a)+b(x+a)$$
$$=(x+a)(x+b)$$

Example: Factorize $x^2+10x+24$.

Solution: We need to determine two numbers a and b such that $a+b=10$ and $ab=24$. Upon observation, we note that $a=4$ and $b=6$, or vice versa. So

$$\begin{aligned}x^2+10x+24 &= x^2+(4+6)x+(4\times6)\\ &= x^2+4x+6x+4\times6\\ &= x(x+4)+6(x+4)\\ &= (x+4)(x+6)\end{aligned}$$

or we can directly write that $x^2+10x+24=x^2+(4+6)x+(4\times6)=(x+4)(x+6)$.

Example: Factorize $x^2-15x+56$.

Solution: The two numbers whose sum is −15 and product is 56 are −8 and −7. So

$$x^2-15x+56=x^2+(-8-7)x+(-8\times-7)=(x-8)(x-7)$$

5. **Factorization of Trinomials of the form Ax^2+Bx+C**

Here we find two numbers a and b such that $a+b=\text{B}$ and $ab=AC$, and then procced as in the previous case.

Example: Factorize $3x^2+11x+10$.

Solution: We are required to find two numbers whose sum is 11 and product is $(3\times10)=30$. The two numbers are 5 and 6. So

$$\begin{aligned}3x^2+11x+10 &= 3x^2+5x+6x+10\\ &= x(3x+5)+2(3x+5)\\ &= (3x+5)(x+2)\end{aligned}$$

Example: Factorize $14x^2-23x+8$.

Solution: The two numbers whose sum is −23 and product is $(14\times8)=112$ are −16 and −7. So

$$\begin{aligned}14x^2-23x+8 &= 14x^2-16x-7x+8\\ &= 2x(7x-8)-1(7x-8)\\ &= (7x-8)(2x-1)\end{aligned}$$

EXAMPLES

1. Find the expression that is equal to the product $(x+6)(x+8)$.

Solution: Using the identity $(x+a)(x+b)=x^2+(a+b)x+ab$, we get

$$(x+6)(x+8)=x^2+(6+8)x+(6)(8)=x^2+14x+48$$

2. Simplify $(x-5)(x-7)$ using an identity.

Solution: Using the identity $(x+a)(x+b)=x^2+(a+b)x+ab$, we get

$$\begin{aligned}(x-5)(x-7) &= [x+(-5)][x+(-7)]\\ &= x^2+[(-5)+(-7)]x+(-5)(-7)\\ &= x^2-12x+35\end{aligned}$$

3. Simplify 113×107 using identities.

Solution: Using the identity $(x+a)(x+b)=x^2+(a+b)x+ab$, we can simplify 113×107 as follows:

$$\begin{aligned} 113 \times 107 &= (100+13)(100+7) \\ &= (100)^2+(13+7)\times 100+(13\times 7) \\ &= 10000+20(100)+91 \\ &= 10000+2000+91=12091 \end{aligned}$$

4. Expand: **(i)** $\left(3a+\frac{2}{3}b\right)^2$ **(ii)** $\left(\frac{2}{3}a+\frac{3}{4}b\right)^2$

Solution: Use the identity $(a+b)^2=a^2+2ab+b^2$ to expand both the expressions as follows:

(i) $$\begin{aligned} \left(3a+\frac{2}{3}b\right)^2 &= (3a)^2+2\times 3a\times\frac{2}{3}b+\left(\frac{2}{3}b\right)^2 \\ &= 9a^2+4ab+\frac{4}{9}b^2 \end{aligned}$$

(ii) $$\begin{aligned} \left(\frac{2}{3}a+\frac{3}{4}b\right)^2 &= \left(\frac{2}{3}a\right)^2+2\times\frac{2}{3}a\times\frac{3}{4}b+\left(\frac{3}{4}b\right)^2 \\ &= \frac{4}{9}a^2+ab+\frac{9}{16}b^2 \end{aligned}$$

5. Write the expanded form of $(3x+5)^3$.

Solution: Consider the identity $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

Put $a=3x$, and $b=5$. Then

$$\begin{aligned} (3x+5)^3 &= (3x)^3+3(3x)^2(5)+(3)(3x)5^2+5^3 \\ &= 27x^3+135x^2+225x+125 \end{aligned}$$

6. Express $9a^2+42ab+49b^2$ as a perfect square.

Solution: $9a^2+42ab+49b^2$

$$\Rightarrow (3a)^2+2\times 3a\times 7b+(7b)^2$$

$$\Rightarrow (3a+7b)^2$$

7. Find the value of $36x^2+84xy+49y^2$ for $x=3$ and $y=6$.

Solution: $36x^2+84xy+49y^2$

$$\begin{aligned} &= (6x)^2+2\times 6x\times 7y+(7y)^2 \\ &= (6x+7y)^2 \\ &= (6\times 2+7\times 6)^2 \qquad (\because x=3,\ y=6) \\ &= (18+42)^2=(60)^2=3600 \end{aligned}$$

8. Find the degree of the following polynomials:

(i) $5x^3 - 3x^2 + 4x - 8$ **(ii)** $6 + 5y^2 - 7y^4$ **(iii)** $2z^5 - 3z^3 + 8z + 1$

Solution: (i) $5x^3 - 3x^2 + 4x - 8$ is a polynomial of degree 3.

(ii) $6 + 5y^2 - 7y^4$ is a polynomial of degree 4.

(iii) $2z^5 - 3z^3 + 8z + 1$ is a polynomial of degree 5.

9. Find HCF of $(4x-5)(2x+3)$ and $(4x-5)^2(3x+7)$.

Solution: HCF of $(4x-5)(2x+3)$ and $(4x-5)^2(3x+7) = (4x-5)$

10. Find LCM of $(4x-5)(2x+3)$ and $(4x-5)^2(3x+7)$.

Solution: LCM of $(4x-5)(2x+3)$ and $(4x-5)^2(3x+7) = (4x-5)^2(2x+3)(3x+7)$

11. Factorize $k^2 - 18k + 81$.

Solution: The two numbers whose sum is –18 and product is 81 are –9 and –9. So

$$k^2 - 18k + 81 = k^2 + (-9-9)p + (-9 \times -9)$$
$$= (k-9)(p-9)$$
$$= (k-9)^2$$

RECALL

1. An identity is an equality which is true for all values of the variables in it.
2. Some commonly used products and identities to simplify expressions are:

   (i) $(x+a)(x+b) = x^2 + (a+b)x + ab$ **(ii)** $(a+b)^2 = a^2 + 2ab + b^2$

   (iii) $(a-b)^2 = a^2 - 2ab + b^2$ **(iv)** $(a+b)(a-b) = a^2 - b^2$

   (v) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$ **(vi)** $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

   (vii) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
3. An exponent is a real number that expresses the power to which a quantity must be raised or lowered.
4. Laws of exponents are the rules that help simplify expressions involving exponents.
5. An algebraic expression consisting of two or more algebraic terms is called a polynomial.
6. Degree of a polynomial in one variable is defined as the highest power of the variable in the polynomial.
7. Degree of a polynomial in two variables is the highest sum of the powers of all the variables in each term of the polynomial.
8. Based on the degree of a polynomial, it can be classified as linear, quadratic, cubic, or constant.
9. LCM of two or more polynomials = LCM (coefficients) × LCM (variables)
10. HCF of two or more polynomials = HCF (coefficients) × HCF (variables)
11. Factorization of polynomials can be done by taking common factors, by grouping, by using idenities, or by using the procedures defined for the forms $x^2 + px + q$ and $Ax^2 + Bx + C$.

DESCRIPTIVE QUESTIONS

I. VERY SHORT ANSWER QUESTIONS

1. Simplify 108×96 using an identity.
2. Evaluate $(105)^2$ by using the expansion of $(a+b)^2$.
3. Evaluate $(47)^2$ by using the expansion of $(a-b)^2$.
4. Find the following products using the identity $(a+b)(a-b)=a^2-b^2$.
   (i) $\left(\frac{3}{5}b+\frac{2}{3}c\right)\left(\frac{3}{5}b-\frac{2}{3}c\right)$
   (ii) $(x-2y)(x+2y)$
5. Evaluate (9.6×10.4) using the identity $(a+b)(a-b)=a^2-b^2$.
6. Find the HCF of $12ab^3$ and $15a^2b^2$.
7. Factorize: $14ab^3+21a^2b^2$
8. Write the factor form of $4a(a^3+1)-8(a^3+1)$.
9. Factorize: $x^2+xy+8x+8y$
10. Factorize: $ax-(ax+by)^2+a^2x+aby+by$
11. Find the value of $4x^3+3x^2+10x+11$ for $x=-3$.
12. Find the value of $m^3+n^3-p^3-3mnp$ when $m=4, n=-3$, and $p=2$.
13. If $a+b=100$ and $a-b=50$, then find the value of $4ab$.
14. Factorize: $a^2+8a+16$
15. Expand: $\left(\frac{2}{3}x-6\right)^3$

II. SHORT ANSWER QUESTIONS

16. Identify the following polynomials based on their degree.
    (i) $3x+\frac{3}{x}$
    (ii) $2+5x^{3/2}+7x^2$
    (iii) $z+3\sqrt{z}+9$
17. Find the degree of the following polynomials and classify them.
    (i) $2x^2y^3-3xy^2+5x^3y^3$
    (ii) $3ab^2-4a\sqrt{b}+5b^3$
    (iii) $4xy-3xy^2+\frac{3x}{y}$
18. Factorize: $3a^3+24b^3$
19. Factorize: $x^2+7x+12$
20. Factorize: $x^2-9x+20$
21. Expand the following using the identity $(a-b)^2=a^2-2ab+b^2$.
    (i) $\left(\frac{1}{3}x-3y\right)^2$ (ii) $\left(3x-\frac{1}{2x}\right)^2$
22. Expand $(p+q+2r)^2$ using an identity.
23. What must be added to $4x^2+20xy$ to make it a perfect square?
24. Write $16m^2-24mn$ as a perfect square expression by making the necessary additions.

III. LONG ANSWER QUESTIONS

25. Simplify: $\frac{2^{-3}\times 4^{-2}\times 3^6}{2^2\times 9\times 3^{-5}}$
26. Simplify: $\frac{(2^{-3})^2\times(3^2)^{-3}\times(5^{-4})^2}{(2^{-2})^5\times(3^3)^{-2}\times(5^{-3})^4}$
27. Simplify:
$$\left\{\left[\left(\frac{2}{5}\right)^2\right]^{-3}\div\left(\frac{3}{5}\right)^{-5}\right\}\times\left(\frac{27}{16}\right)^{-1}\times\left(\frac{9}{4}\right)^{-2}\times 3^{-1}$$
28. Find n such that $\left(\frac{3}{5}\right)^4\times\left(\frac{3}{5}\right)^3=\left(\frac{3}{5}\right)^{2n-1}$.

IV. FILL IN THE BLANKS

29. An __________ is an equality or an algebraic equation which is true for all values of the variables in it.

30. The product $(x+3)(x+6)$ is equal to ________.

31. The polynomial x^3+2x^2+4x-3 is a ________ polynomial.

32. The product $\left(\frac{7}{11}x+\frac{4}{7}y\right)\left(\frac{7}{11}x-\frac{4}{7}y\right)$ is equal to ________.

33. To make $4m^2 - 20mn$ a perfect square, ________ must be added to it.

34. In 5^2, ________ is called the base and ________ is called the exponent.

35. The HCF of $12ab$, $4a^2b$, $13a^2b^3$ is ________.

36. $\frac{-3}{7}\times\frac{-3}{7}\times\frac{-3}{7}$ can be written in the power notation as ________.

37. $\left(\frac{3}{2}\right)^{-3} =$ ________

38. The expression $(8x+6)(3x+2)-7$ is not in ________ form.

39. The degree of the polynomial $16x^3-12x^2+5x-7$ is ________.

V. TRUE OR FALSE

40. $\left\{\left(\frac{-5}{3}\right)^3\right\}^2$ is equal to $\frac{6125}{729}$.

41. $\frac{-27}{64}$ can be expressed in exponential form as $\left(\frac{-3}{4}\right)^3$.

42. If b^2 is added to $4a^2 + 8ab$, then it becomes a perfect square.

43. $6a^2b^3+3a^3b^2-2a^4b^3$ is a polynomial in a and b of degree 2.

44. HCF of $15xy^3$ and $35x^2y^2$ is $5xy^2$.

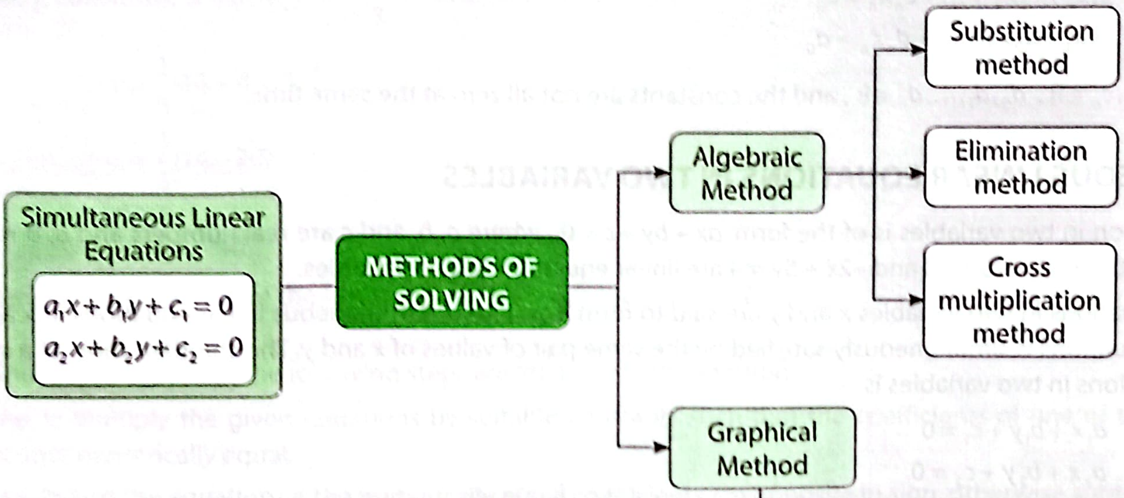
45. LCM of two or more polynomials = (LCM of numerical coefficients) × (each factor raised to lowest power)

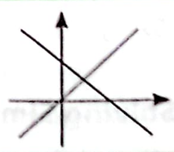
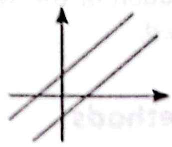
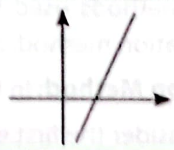
46. The factor form of $14x^2-23x+8$ is $(7x-8)(2x+1)$.

47. The expansion of $(2x-3)^3$ is $8x^3-36x^2+54x+27$.

3.02 Linear Equations in Two Variables

MIND MAP



Type of Solution	Condition	Graphical Representation
Unique solution (consistent and independent)	Intersecting lines	
No solution (inconsistent)	Parallel lines	
Infinite solutions (consistent and dependent)	Coincident lines	

INTRODUCTION

As discussed earlier, polynomials of degree one are called linear polynomials. If we equate these polynomials to zero, they become linear equations. We do this to solve polynomials which may represent real world scenarios like determining the yearly savings of a person when the spending trends are nearly the same every month or finding the number of different kind of fruits to be packed in a dozen baskets when the capacity of one basket is known etc.

Linear Equations

The general form of a linear equation is $c_1x_1 + c_2x_2 + \ldots + c_nx_n = c_0$, where $x_1, x_2, \ldots, x_n$ are variables, $c_0, c_1, \ldots, c_n$ are constants, and n is the number of variables in the equation.

System of Linear Equations or Simultaneous Linear Equations

A collection of linear equations that simultaneously hold true for the same values of the variables is called a system of linear equations. The standard form of a system of linear equations is

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = c_0$$
$$d_1x_1 + d_2x_2 + \ldots + d_nx_n = d_0$$

where $c_0, c_1, \ldots, c_n \in R$, $d_0, d_1, \ldots, d_n \in R$, and the constants are not all zero at the same time.

SIMULTANEOUS LINEAR EQUATIONS IN TWO VARIABLES

A linear equation in two variables is of the form $ax + by + c = 0$, where a, b, and c are real numbers and $a, b \neq 0$. For instance, equations $11x + 5y = 8$ and $-2x + 5y = 4$ are linear equations in two variables.

Two linear equations in two variables x and y are said to form a system of simultaneous linear equations in x and y if each of the equations is simultaneously satisfied by the same pair of values of x and y. The standard form of a system of linear equations in two variables is

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in R$ and $a_1, a_2, b_1, b_2 \neq 0$.

Example: If x and y are the variables, then a system of linear equations is

$$3x + 2y = 7$$
$$5x - y = 3$$

Methods of Solving Simultaneous Linear Equations

To obtain a solution of the system $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, either algebraic methods or graphical methods are used.

Algebraic Methods

The algebraic methods used to solve a system of simultaneous linear equations in two variables are substitution method, elimination method, and cross multiplication method (*to be learnt in higher grades*).

1. **Substitution Method:** In this method, the following steps are followed:

   Step 1: Consider the first equation and express y in terms of x.

   Step 2: Substitute this value of y in the second equation and simplify for x.

   Step 3: Substitute the value of x in the relation ob tained in Step 1 to obtain the value of y.

   Example: Solve the system $4x + 3y = 14$, $3x + 2y = 11$.

   Solution: The given equations are

   $$4x + 3y = 14 \quad \ldots(1)$$
   $$3x + 2y = 11 \quad \ldots(2)$$

   From equation (1), we get $4x + 3y = 14 \Rightarrow y = \dfrac{14 - 4x}{3}$.

Substitute $y=\frac{14-4x}{3}$ in equation (2) and simplify for *x*.

$$3x+2\left(\frac{14-4x}{3}\right)=11 \Leftrightarrow 9x+2(14-4x)=33$$

$$\Leftrightarrow 9x+28-8x=33$$

$$\Leftrightarrow \quad x=33-28=5$$

Finally, substitute $x=5$ in $y=\frac{14-4x}{3}$ and solve for *y*.

$$y=\frac{1}{3}(14-4\times5)$$

$$=\frac{1}{3}(14-20)$$

$$=\frac{1}{3}(-6)=-2$$

Therefore, the solution of the given system is $x=5, y=-2$ or $(5,-2)$.

2. **Elimination Method:** The following steps are followed in this method:

   Step 1: Multiply the given equations by suitable constants such that the coefficients of one of the unknowns become numerically equal.

   Step 2: Add the equations if the numerically equal coefficients are opposite in sign, otherwise subtract them.

   Step 3: Solve the resultant equations which give the value of one of the unknowns.

   Step 4: Substitute this value in any of the given equations and solve to find the value of the other unknown.

Example: Solve the system $4x+7y=9$, $10x+9y=31$.

Solution: In this system, let us convert the coefficients of *x* into numerically equivalent coefficients.

The coefficients of *x* are 4 and 10. Since LCM (4, 10) = 20, we multiply the first equation by 5 and the second equation by 2, as shown below:

$$(4x+7y=9) \quad \times 5$$

$$(10x+9y=31) \quad \times 2$$

$$20x+35y=45$$

$$20x+18y=62$$

Subtract the second equation from the first to obtain the value of *y*.

$$20x+35y=45$$

$$20x+18y=62$$

$$(-)\,(-) \quad (-)$$

$$17y=-17$$

$$y=-1$$

Finally, substitute $y=-1$ in $4x+7y=9$ and solve for *x*.

$$4x+7\times(-1)=9 \Leftrightarrow 4x-7=9$$

$$\Leftrightarrow 4x=16$$

$$\Leftrightarrow x=4$$

Therefore, the solution of the given system is $x=4$, $y=-1$ or $(4,-1)$.

Key point

Special Case: If the system is of the form $ax+by=c_1$, $bx+ay=c_2$ and the coefficients a and b are large, we try to obtain a simpler system by first adding these two equations and then subtracting them. The system of two equations thus obtained has the same solution set as the original one.

Consistency and Inconsistency of a System of Equations

Note that a system of equations can have one solution, infinitely many solutions, or no solution at all. If a system has at least one solution, it is said to be a consistent system. If a system has no solution at all, it is said to be an inconsistent system.

The conditions that help determine the consistency or inconsistency of a given system of linear equations are:

(i) If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the system is said to be consistent and independent with a unique solution.

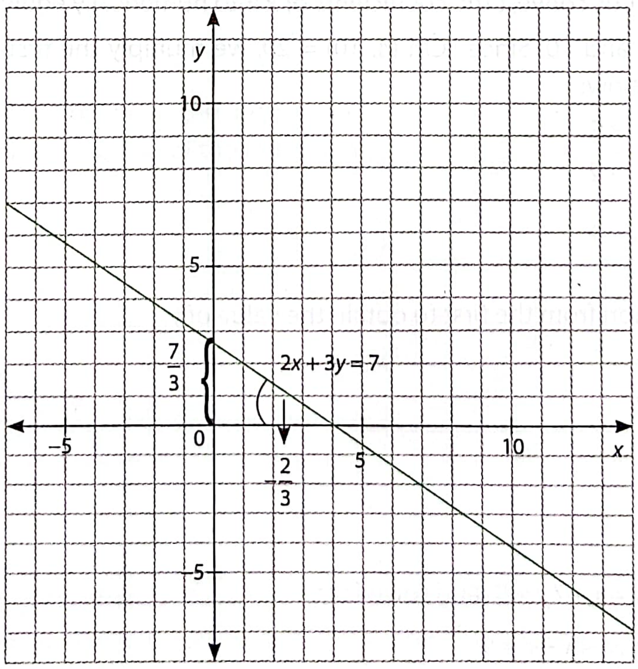
(ii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the system is said to be inconsistent, that is, it has no solution at all.

(iii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the system is said to be consistent and dependent with infinitely many solutions.

Graphical Methods

The standard form of a linear equation is $ax + by + c = 0$. When plotted on a graph, the nature of the curve is a straight line that may or may not cut the coordinate axes or pass through the origin (0,0).

For instance, if the equation $2x+3y=7$ is plotted on a graph, it is represented as a straight line which has a slope (the angle of inclination with the x-axis) of $-\frac{2}{3}$ and a y-intercept (the point at which the line cuts the y-axis) of $\frac{7}{3}$.



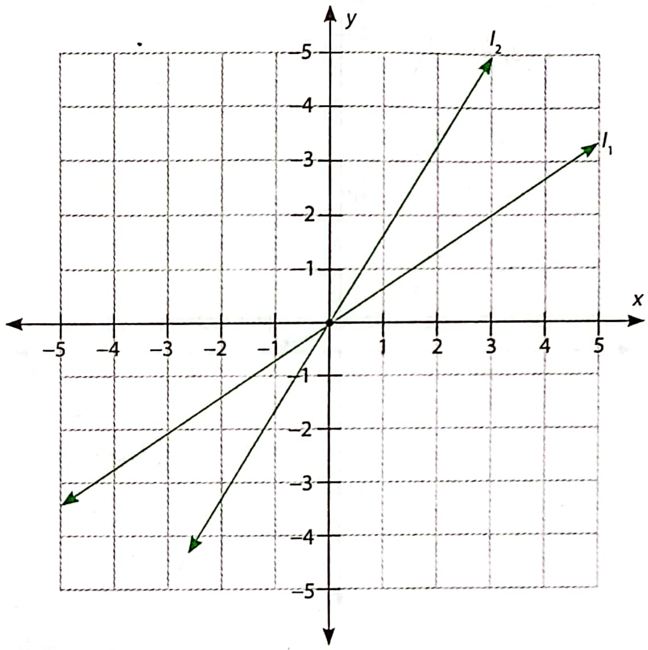
Slope: The equation $ax+by+c=0$ can be rewritten as $y=-\frac{a}{b}x-\frac{c}{b}$ which is equivalent to the form $y=mx+d$, where m is the slope of the line and d is the y-intercept of the line.

So, for the line $ax+by+c=0$, its slope is given by $m=-\frac{a}{b}$ and the y-intercept is given by $d=-\frac{c}{b}$.

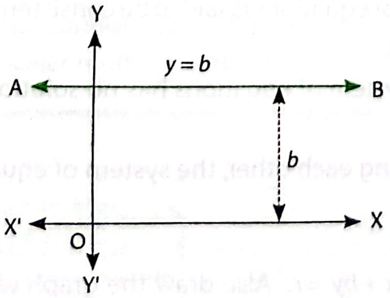
Analyzing Graphs of System of Linear Equations in Two Variables

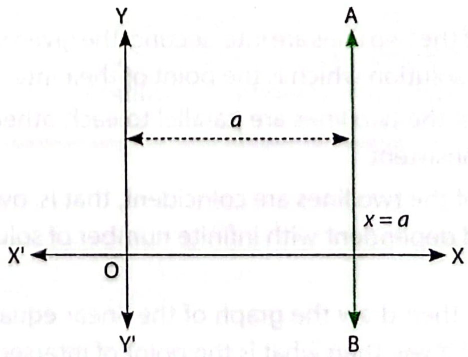
A straight line can either pass through the origin or can be parallel to one of the coordinate axes or can be inclined at a certain angle to one of the coordinate axes.

1. **Lines passing through the origin:** If two or more lines pass through the origin, then their solution is (0, 0). These equations can be represented graphically as follows:

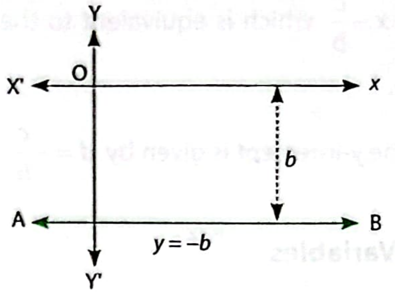


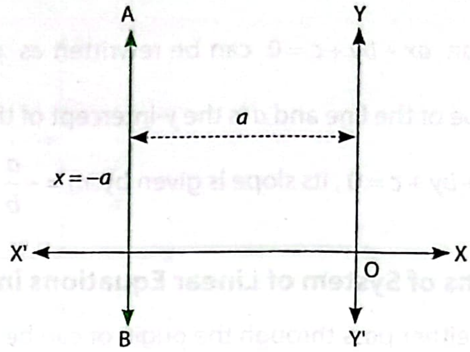
2. **Lines parallel to coordinate axes:** The line of the form $y = b$ is represented graphically by a line parallel to the x-axis at a distance of b units, whereas the line of the form $x = a$ is represented graphically by a line parallel to the y-axis at a distance of a units.



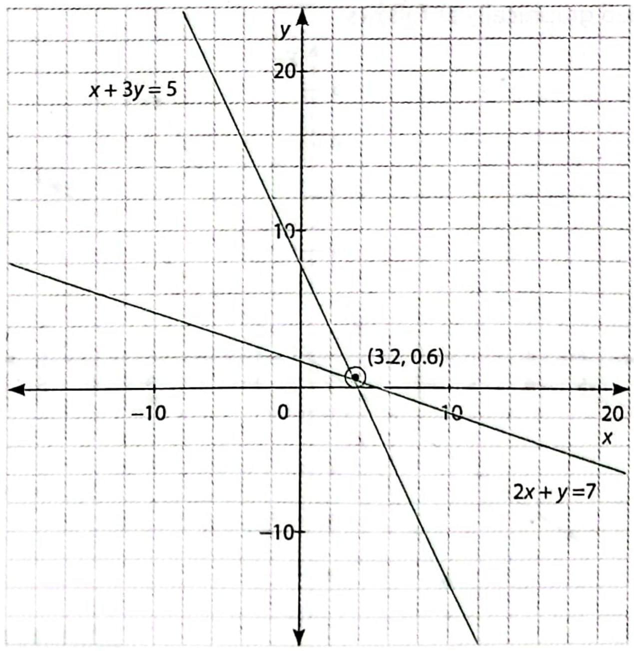


Depending on the sign negative or positive, of the value of b, the line $y = b$ lies below or above the x-axis. Same can be commented for the value of a, that is, depending on the sign of the value of a, the line $x = a$ lies on the left or the right of the y-axis.





3. **Intersecting lines:** The lines $x+3y=5$ and $2x+y=7$, that intersect at the point (3.2, 0.6), can be represented graphically as follows:



Consistency and Inconsistency of a System of Equations

Graphical analysis of the consistency or inconsistency of a system of linear equations is simpler than that by algebraic methods.

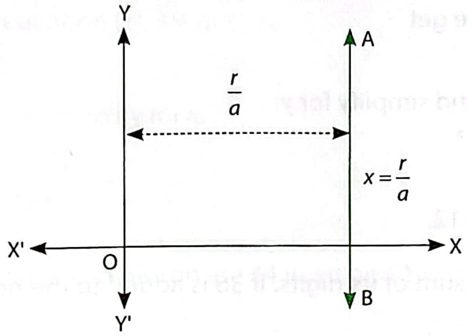
(i) If the graphs of the two lines are intersecting, the given system of equations is said to be consistent and independent with a unique solution which is the point of their intersection.

(ii) If the graphs of the two lines are parallel to each other, the system of equations has no solution and is therefore, said to be inconsistent.

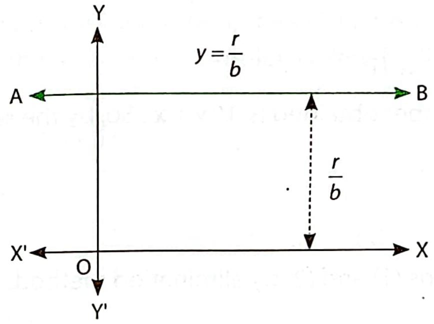
(iii) If the graphs of the two lines are coincident, that is, overlapping each other, the system of equations is said to be consistent and dependent with infinite number of solutions.

Example: If $b=0$, then draw the graph of the linear equation $ax+by=r$. Also draw the graph when $a=0$. Do the two lines intersect? If yes, then what is the point of intersection?

Solution: If $b=0$, then the linear equation $ax+by=r$ becomes $ax=r \Rightarrow x=\frac{r}{a}$ which is a line parallel to the y-axis at a distance of $\frac{r}{a}$ units on the right side of the y-axis.



If $a=0$, then the linear equation $ax+by=r$ becomes $by=r \Rightarrow y=\frac{r}{b}$ which is a line parallel to the x-axis at a distance of $\frac{r}{b}$ units above the x-axis.



Clearly, the lines $x=\frac{r}{a}$ and $y=\frac{r}{b}$ intersect at the point $\left(\frac{r}{a},\frac{r}{b}\right)$.

MISCONCEPTION

Both parallel and coinciding lines have infinitely many solutions.

FACT: Parallel lines have no solution, whereas coinciding lines have infinitely many solutions.

Parallel lines are of the form $ax+by=c_1$, $ax+by=c_2$, that is, they differ only in the constant term which is the condition for no solution. Coincident lines are of the form $ax+by=c$, $akx+bky=ck$, where $k \neq 0$, that is, one equation is a scalar multiple of the other which satisfies the condition for infinitely many solutions.

EXAMPLES

1. The sum of two numbers is 57 and their difference is 33. Find the numbers.

Solution: Let the required numbers be x and y. Then

$x+y=57$ …(1)

$x-y=33$ …(2)

On adding equations (1) and (2), we get

$$2x = 90 \Rightarrow x = 45$$

Substitute $x = 45$ in equation (1) and simplify for y.

$$x + y = 57 \Rightarrow 45 + y = 57$$
$$\Rightarrow y = 12$$

Therefore, the numbers are 45 and 12.

2. A two-digit number is 4 times the sum of its digits. If 36 is added to the number, its digits are reversed. Find the number.

Solution: Let the tens and the units digit of the number be x and y, respectively. Then, the number is $10x + y$ and the sum of its digit is $x + y$.

By the first condition, we get

$$10x + y = 4(x + y)$$
$$\Rightarrow 10x + y = 4x + 4y$$
$$\Rightarrow 6x - 3y = 0 \qquad \text{...(1)}$$

On reversing the digits, the new number obtained is $10y + x$. So, by the second condition, we get

$$(10x + y) + 36 = 10y + x$$
$$\Rightarrow 9x - 9y = -36$$
$$\Rightarrow x - y = -4 \qquad \text{...(2)}$$

We now solve the system of equations (1) and (2) by elimination method.

$$6x - 3y = 0$$
$$(x - y = -4) \quad \times 3$$

$$6x - 3y = 0$$
$$3x - 3y = -12$$
$$(-) \quad (+) \quad (+)$$

$$3x = 12$$
$$x = 4$$

Substitute $x = 4$ in equation (1) and simplify for y.

$$6x - 3y = 0 \Rightarrow 6 \times 4 = 3y$$
$$\Rightarrow y = 8$$

Therefore, the original number is $10x + y = 10 \times 4 + 8 = 48$.

3. Ten years ago, a man was 12 times as old as his son and 10 years hence, the man will be twice as old as his son. Find their present ages.

Solution: Let the present ages of the man and his son be x years and y years, respectively.

10 years ago, the man was 12 times the age of his son. So,

$$x - 10 = 12(y - 10)$$
$$\Rightarrow x - 10 = 12y - 120$$
$$\Rightarrow x - 12y = -110 \qquad \text{...(1)}$$

After 10 years, the man's age will be 2 times the age of his son. So,

$$x + 10 = 2(y + 10)$$
$$\Rightarrow x + 10 = 2y + 20$$
$$\Rightarrow x - 2y = 10 \qquad \text{...(2)}$$

On subtracting equation (2) from equation (1), we get

$$-10y = -120 \Leftrightarrow y = 12$$

Substitute $y = 12$ in equation (2) and simplify for x.

$$x - 2y = 10 \Rightarrow x - 2\times12 = 10$$
$$\Rightarrow x - 24 = 10$$
$$\Rightarrow x = 34$$

Therefore, the present ages of the man and his son are 34 years and 12 years, respectively.

4. Solve the system $19x - 17y = 55, 17x - 19y = 53$.

Solution: The given equations are

$$19x - 17y = 55 \quad \ldots(1)$$
$$17x - 19y = 53 \quad \ldots(2)$$

If we use any of the algebraic methods on this system, the calculations will become complex and might lead to some errors as well. So, we reduce this system to a simpler system by first adding the given equations and then subtracting them.

$$(19x - 17y) + (17x - 19y) = 55 + 53$$
$$36x - 36y = 108$$
$$x - y = 3 \quad \ldots(3)$$
$$(19x - 17y) - (17x - 19y) = 55 - 53$$
$$2x + 2y = 2$$
$$x + y = 1 \quad \ldots(4)$$

We now solve the system of equations (3) and (4) using elimination method.

$$x - y = 3$$
$$x + y = 1$$
$$\underline{(-)\ (-)(-)}$$
$$-2y = 2$$
$$y = -1$$

Substitute $y = -1$ in equation (3) and simplify for x.

$$x - y = 3 \Rightarrow x + 1 = 3$$
$$\Rightarrow x = 2$$

Therefore, the solution of the given system is $x = 2, y = -1$ or $(2, -1)$.

5. Give the condition for which a given system of linear equations in two variables has no solution.

Solution: Consider the system $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the system has no solution and is said to be inconsistent.

6. Give the condition for which a given system of linear equations in two variables has infinitely many common solutions.

Solution: Consider the system $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the system is consistent and dependent with infinitely many solutions.

7. Consider the following system of linear equations in two variables:

$$x - 2y = -4$$
$$-3x + 6y = 0$$

Why do they not have a solution? Also plot the graph.

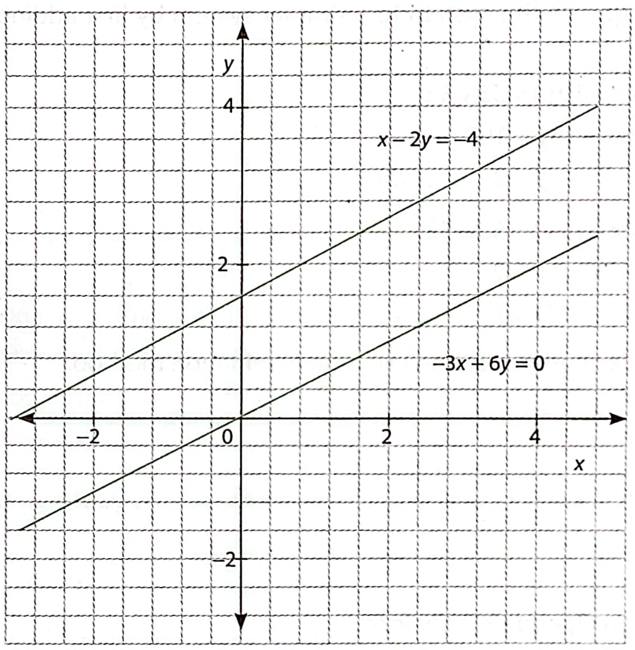
Solution: For the given system, let us check the ratio of their coefficients.

$$\frac{a_1}{a_2} = \frac{1}{-3} = -\frac{1}{3}$$

$$\frac{b_1}{b_2} = \frac{-2}{6} = -\frac{1}{3}$$

$$\frac{c_1}{c_2} = \frac{-4}{0}$$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, the given system has no solution. The graph of the system is given below:



8. Determine the equation of a line which is parallel to the x-axis at a distance of 8 units above the x-axis.

Solution: The equation of a straight line parallel to the x-axis is of the form $y = k$, where k is the distance of the line from the x-axis.

Since the distance is 8 units above the x-axis, the equation becomes $y = 8$.

9. Find the equation of a line which is parallel to the y-axis at a distance of 15 units on the right side of the y-axis.

Solution: The equation of a straight line parallel to the y-axis is of the form $x = k$, where k is the distance of the line from the y-axis.

Since the distance is 15 units on the right of the y-axis, the equation becomes $x = 15$.

10. Solve the system $3x + 5y = 3$, $4x + 3y = 15$.

Solution: The given system of equations is

$$3x + 5y = 3 \qquad \ldots(1)$$
$$4x + 3y = 15 \qquad \ldots(2)$$

We solve this system by elimination method.

$$(3x+5y=3) \quad \times 4$$
$$(4x+3y=15) \quad \times 3$$
$$12x+20y=12$$
$$12x+9y=45$$
$$(-)\ (-) \quad (-)$$
$$11y=-33$$
$$y=-3$$

Substitute $y=-3$ in equation (1) and solve for x.

$$3x+5y=3 \Rightarrow 3x+5(-3)=3$$
$$\Rightarrow 3x-15=3$$
$$\Rightarrow 3x=18$$
$$\Rightarrow x=6$$

Therefore, the solution of the given system is $x=6$, $y=-3$ or $(6, -3)$.

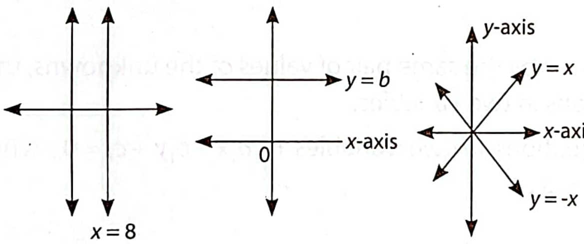
RECALL

1. An equation of the form $ax+by+c=0$, where a, b, and c are real numbers is called a linear equation in two variables x and y.
2. When two linear equations in two unknowns are satisfied by the same pair of values of the unknowns, then these equations are called simultaneous linear equations in two variables.
3. General form of a system of simultaneous linear equations in two variables is $a_1x+b_1y+c_1=0$, where $a_1 \neq 0, b_1 \neq 0$ and $a_2x+b_2y+c_2=0$, where $a_2 \neq 0, b_2 \neq 0$.
4. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ or the graphs of two lines intersect, then the system is said to be consistent and independent with a unique solution.
5. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ or the graphs of two lines are parallel, then the system is said to be inconsistent, that is, it has no solution.
6. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ or the graphs of two lines are coincident, then the system is said to be consistent and dependent with infinite number of solutions.
7. The equation $x=a$ (or $x=-a$) represents a line parallel to the y-axis at a distance of a units on the right (or left) side of the y-axis.
8. The equation $y=b$ (or $y=-b$) represents a line parallel to the x-axis at a distance of b units above (or below) the x-axis.

DESCRIPTIVE QUESTIONS

I. VERY SHORT ANSWER QUESTIONS

1. What are the algebraic methods used to solve linear equations in two variables?
2. If $a \neq 0, b = 0$, and $c \neq 0$, then the equation $ax + by + c = 0$ becomes $ax + c = 0$ or $x = -\frac{c}{a}$. Discuss the nature of the graph.
3. When does a system of linear equations in two variables have a unique solution?
4. For no solution, analyze the nature of graph of a system of equations in two variables.
5. Draw the graph of $x = \frac{-5}{4}$.
6. When $a \neq 0$, $b = 0$, and $c = 0$, then what happens to the equation $ax + by + c = 0$? Analyze the nature of the graph.
7. Analyze the following graph:



8. When is a system of linear equations in two variables said to be dependent with infinitely many common solutions?

II. SHORT ANSWER QUESTIONS

9. Show that $x = 5, y = -2$ is a solution of the system $4x + 3y = 14, 9x - 5y = 55$.
10. Show that $x = 3, y = 2$ is not a solution of the system $4x - 3y = 6, 5x + 2y = 17$.
11. Solve the system $2x - 3y = 7, x + y = 1$ by using the method of substitution.
12. The larger of two supplementary angles exceeds the smaller by 18°. Find their measures.
13. Find the equation of a line parallel to the x-axis such that it lies at a distance of 7 units below the x-axis.
14. Determine the equation of a line parallel to the y-axis at a distance of 4 units on the left of the y-axis.
15. Atul has a farm where he domesticates hens and cows. If the total number of heads is 64 and the total number of feet is 196, find the number of hens and cows that Atul has.

III. LONG ANSWER QUESTIONS

16. Consider the following system of linear equations in two variables:

$5x + 2y = 4$

$-2x + y = 11$

Determine the solution and plot a graph for the same.

17. For the system $8x + 7y = 38, 3x - 5y = -1$, analyze the type of solution. Also plot the graph.
18. For the following system, determine the solution and plot a graph.

$3y = 18 - x$

$3x - 7y = -10$

19. Solve the system $x + y = 13, x - y = 3$. Plot the solution on a graph.
20. Discuss the nature of solution of two lines in a plane.

IV. FILL IN THE BLANKS

21. A linear equation in two variables has two variables, each of degree ______.
22. A pair of values of x and y satisfying each of the equations in a given system of two simultaneous equations in x and y is called a ________ of the system.
23. In the ______ method, the coefficients of any one variable in both the equations are made equal by multiplying the equations with suitable non-zero numbers.
24. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the system of linear equations is said to be consistent with ______ solution.

V. TRUE OR FALSE

25. An equation is said to be linear in two variables if it can be written in the form $ax + by + c = 0$.

26. The coefficients of x and y are zero in $ax + by + c = 0$.

27. If two lines are parallel, the system is consistent and there exists a unique solution which is the point of intersection of the two lines.

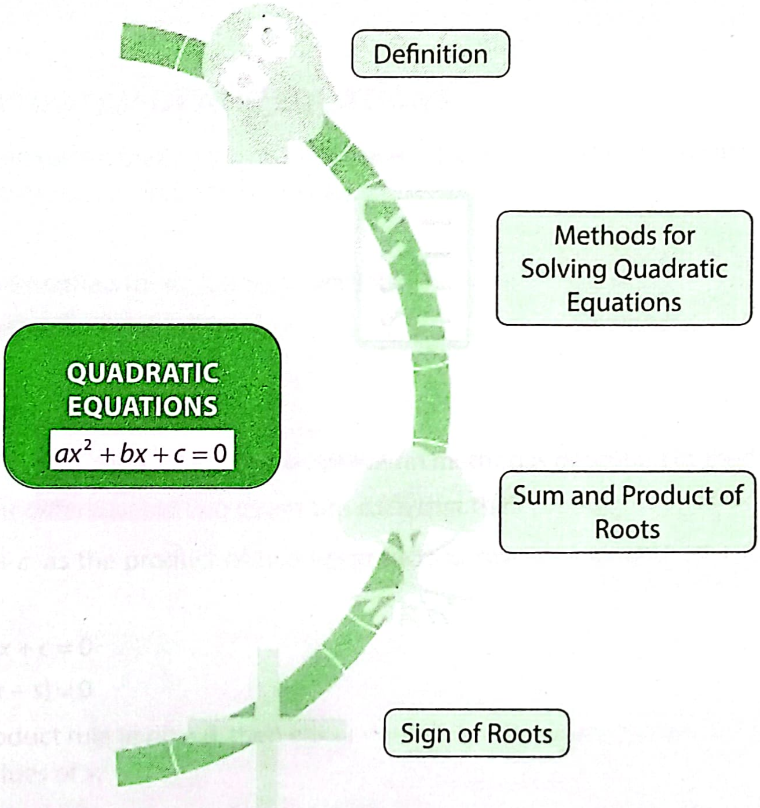
28. If the numerically equal coefficients in the elimination method are opposite in sign, then add the two equations, otherwise subtract them.

29. The equation $y = b$ represents a line that is parallel to the y-axis at a distance of b units on the right of the y-axis.

30. If the sum of two numbers is 60 and their difference is 14, then the numbers are 27 and 33.

3.03 Introduction to Quadratic Equations

MIND MAP



INTRODUCTION

Quadratic equations find applications in almost every aspect of our lives. Be it calculating the area of a room or finding the maximum height of a ball thrown by a kid or determining the speed of a ship on sea, quadratic equations are involved.

The word "quadratic" is derived from the Latin word *quadratus* meaning "squared". A quadratic expression is a second-degree polynomial, that is, the highest power of the variable in the polynomial is two (hence the word *squared*).

QUADRATIC EQUATIONS

A quadratic equation is a polynomial equation of second degree. Its general form is

$$ax^2 + bx + c = 0$$

where $a \neq 0$ (in case $a = 0$, the equation becomes linear). The letters a, b, and c are called coefficients; a is called the quadratic coefficient, b is called the linear coefficient, and c is called the constant coefficient, also called the free term or the constant term. For instance, $5x^2+7x+3=0$, $4x^2+2=0$, and $3x^2+8x+4=8$ are all quadratic equations.

A quadratic equation will always have two roots. These roots can either be equal or distinct, or real or imaginary.

Key point

An equation of the form $ax^2+bx+d=e$, where a, b, d, and e are known constants, can be reduced to the general form of quadratic equation as $ax^2+bx+d=e \Rightarrow ax^2+bx+c=0$, where $c=d-e$.

METHODS FOR SOLVING QUADRATIC EQUATIONS

Solving a quadratic equation means finding the roots of the equation, that is, the values that satisfy the given quadratic equation. There are three methods used for this purpose.

(i) Factorization method

(ii) Completing the square method (or square root method)

(iii) Quadratic formula method

Factorization Method

Consider the equation $ax^2+bx+c=0$, $a \neq 0$. The factorization method is described in the following steps:

Step 1: Write b as a sum (or difference) of two quantities such that their product is equal to ac.

Step 2: Express ax^2+bx+c as the product of two linear factors, say $(px+q)$ and $(rx+s)$, where $p, q, r, s \in R$ and $p \neq 0$, $r \neq 0$.

$$ax^2+bx+c=0$$
$$(px+q)(rx+s)=0$$

Step 3: Using the zero-product rule (if $ab = 0$, then either $a = 0$ or $b = 0$), equate the linear factors $(px+q)$ and $(rx+s)$ to 0 and determine the values of x.

$$(px+q)(rx+s)=0 \Rightarrow px+q=0 \text{ or } rx+s=0$$
$$\Rightarrow x=-\frac{q}{p} \text{ or } x=-\frac{s}{r}$$

Therefore, the roots of the quadratic equation become $-\frac{q}{p}$ and $-\frac{s}{r}$.

Example: Find the roots of $2x^2-x-3=0$ by factorization method.

Solution: In $2x^2-x-3=0$, $a=2$, $b=-1$ and $c=-3$. Then, b can be written as the difference of 3 and 2 as $3\times-2=-6=ac$. So

$$2x^2-x-3=0$$
$$2x^2-(3-2)x-3=0$$
$$(2x^2-3x)+(2x-3)=0$$
$$x(2x-3)+1(2x-3)=0$$
$$(2x-3)(x+1)=0$$

$$\Rightarrow 2x-3=0 \text{ or } x+1=0$$

$$\Rightarrow x=\frac{3}{2} \text{ or } x=-1$$

Hence, the roots are $\frac{3}{2}$ and -1.

Completing the Square Method

Consider the equation $x^2+8x+4=0$. If factorization method is applied, we observe that 8 cannot be expressed as a sum (or difference) of two numbers whose product is 4. So, the factorization method fails here. Let us see how completing the square method proves helpful in this situation.

Consider the equation $ax^2+bx+c=0$, $a \neq 0$.

Step 1: Divide the equation by a to convert the quadratic coefficient into 1.

$$ax^2+bx+c=0 \Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Step 2: Add and subtract $\left(\frac{b}{2a}\right)^2$, that is, square of half the coefficient of x, on the left-hand side. Our motive is to convert the LHS into a perfect square.

$$x^2+\frac{b}{a}x+\frac{c}{a}+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2=0$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2=0$$

$$\left[x^2+2\times\frac{b}{2a}\times x+\left(\frac{b}{2a}\right)^2\right]+\frac{c}{a}-\left(\frac{b}{2a}\right)^2=0$$

$$\left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

Step 3: Take square root on both the sides to obtain two linear equations. On solving these equations, two roots of the quadratic equation are obtained.

Note that since square root is being taken in the third step, this method is also known as the square root method.

Example: Solve $x^2+8x+4=0$.

Solution: Since the quadratic coefficient is already 1, we move straight to step 2.

Add and subtract $\left(\frac{8}{2}\right)^2=16$ on LHS to complete the square.

$$x^2+8x+4=0$$

$$x^2+8x+4+16-16=0$$

$$x^2+8x+16-16+4=0$$

$$(x-4)^2-12=0$$

$$(x-4)^2=12$$

$$x-4=\pm\sqrt{12}$$

$$x=4\pm\sqrt{12}$$

Therefore, the roots are $4+\sqrt{12}$ and $4-\sqrt{12}$.

Example: Solve $2x^2+7x+3=0$.

Solution: Since the quadratic coefficient is not 1, we first divide the given equation by 2.

$$2x^2+7x+3=0 \Rightarrow x^2+\frac{7}{2}x+\frac{3}{2}=0$$

Add and subtract $\left(\frac{7}{2\times2}\right)^2=\frac{49}{16}$ on LHS to complete the square.

$$x^2+\frac{7}{2}x+\frac{3}{2}+\frac{49}{16}-\frac{49}{16}=0$$

$$x^2+\frac{7}{2}x+\frac{49}{16}+\frac{3}{2}-\frac{49}{16}=0$$

$$\left(x^2+2\times\frac{7}{4}\times x+\frac{49}{16}\right)-\frac{25}{16}=0$$

$$\left(x+\frac{7}{4}\right)^2=\frac{25}{16}$$

$$x+\frac{7}{4}=\pm\frac{5}{4}$$

$$x=-\frac{1}{2},-3$$

Therefore, the roots are $-\frac{1}{2}$ and -3.

Example: Solve $2x^2+x-4=0$.

Solution: Proceed as follows:

$$2x^2+x-4=0$$

$$x^2+\frac{x}{2}-2=0$$

$$x^2+\frac{x}{2}-2+\frac{1}{16}-\frac{1}{16}=0$$

$$\left(x^2+\frac{x}{2}+\frac{1}{16}\right)-2-\frac{1}{16}=0$$

$$\left(x+2\times\frac{1}{4}\times x+\frac{1}{16}\right)-\frac{33}{16}=0$$

$$\left(x+\frac{1}{4}\right)^2=\frac{33}{16}$$

$$x+\frac{1}{4}=\pm\frac{\sqrt{33}}{4}$$

$$x=\pm\frac{\sqrt{33}-1}{4}$$

Therefore, the roots are $\frac{\sqrt{33}-1}{4}$ and $\frac{-\sqrt{33}+1}{4}$.

Quadratic Formula Method

This method involves using a general formula, called the quadratic formula, to find the solution of any quadratic equation. Let us first see how this formula is derived.

Consider a quadratic equation $ax^2+bx+c=0$, $a\neq 0$.

$$ax^2+bx+c=0$$
$$ax^2+bx=-c$$
$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Add $\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}$ on both sides to convert the LHS into a perfect square.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$
$$x^2+2\times\frac{b}{2a}\times x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$
$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$
$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Therefore, the roots of the equation $ax^2+bx+c=0$ are given by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, which is called the quadratic formula.

Example: Solve $2x^2-5x-3=0$ using the quadratic formula.

Solution: On comparison with the general form of quadratic equation, we get $a=2$, $b=-5$, and $c=-3$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-3)}}{2(2)}$$
$$=\frac{5\pm\sqrt{25+24}}{4}$$
$$=\frac{5\pm 7}{4}$$
$$=\frac{5+7}{4},\frac{5-7}{4}$$
$$=3,-\frac{1}{2}$$

Therefore, the roots are 3 and $-\frac{1}{2}$.

SUM AND PRODUCT OF ROOTS OF A QUADRATIC EQUATION

Sum of Roots

Let us find the sum of the roots $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

$$x_1 + x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-2b}{2a}$$
$$= \frac{-b}{a}$$

Therefore, sum of the roots of any quadratic equation is always equal to $\frac{-b}{a}$ or in a more general sense,

$$\boxed{\text{sum of roots} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}}.$$

Product of Roots

We now determine the product of the roots of $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

$$x_1 x_2 = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$
$$= \frac{(-b)^2 - \left(\sqrt{b^2-4ac}\right)^2}{4a^2}$$
$$= \frac{\cancel{b^2} - \cancel{b^2} + 4ac}{4a^2}$$
$$= \frac{c}{a}$$

Therefore, product of the roots of any quadratic equation is always equal to $\frac{c}{a}$, or in a more general sense,

$$\boxed{\text{product of roots} = \frac{\text{constant coefficient}}{\text{coefficient of } x^2}}.$$

Sign of Roots

Based on the signs of the sum and the product of roots of a quadratic equation, we can comment on the sign of the roots, that is, determine whether they are positive or negative. The following table illustrates the same:

Sum of Roots	Product of Roots	Sign of Roots
+	+	Both roots are positive
–	+	Both roots are negative
+	–	The roots are of opposite signs and the numerically greater root is positive
–	–	The roots are of opposite signs and the numerically greater root is negative

Solutions of Equations Reducible to Quadratic Form

Equations that are not quadratic in nature but can be reduced to quadratic forms by suitable transformations can also be solved by using any of the preceding methods.

CASE I: $ax^4+bx^2+c=0$

This form can be reduced to a quadratic form by substituting $x^2=y$. The equation becomes $ay^2+by+c=0$. After obtaining the values of *y*, the equation $x^2=y$ determines the values of *x*.

Example: Solve for *y*: $9y^4-29y^2+20=0$

Solution: Substitute $y^2=x$ and simplify.

$9y^4-29y^2+20=0$

$\Rightarrow 9x^2-29x+20=0$

$\Rightarrow 9x^2-9x-20x+20=0$

$\Rightarrow 9x(x-1)-20(x-1)=0$

$\Rightarrow (9x-20)(x-1)=0$

$\Rightarrow x=\frac{20}{9},1$

Therefore, the values of *y* are $\pm\frac{\sqrt{20}}{3}$ and ±1.

CASE II: $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$

For this type, we use the identity $x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$. The equation then becomes $a\left[\left(x+\frac{1}{x}\right)^2-2\right]+b\left(x+\frac{1}{x}\right)$ $+c=0$. Now put $x+\frac{1}{x}=y$ to obtain the quadratic equation $ay^2+by+(c-2a)=0$. Once the values of *y* are determined, the equation $x+\frac{1}{x}=y$ determines the values of *x*.

Example: Solve $8\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)=137$.

Solution: The given equation can be rewritten as $8\left[\left(x+\frac{1}{x}\right)^2-2\right]+2\left(x+\frac{1}{x}\right)=137$.

Let $x+\frac{1}{x}=y$.

$8\left[\left(x+\frac{1}{x}\right)^2-2\right]+2\left(x+\frac{1}{x}\right)=137$

$\Rightarrow 8(y^2-2)+2y=137$

$\Rightarrow 8y^2+2y-153=0$

$\Rightarrow 8y^2+36y-34y-153=0$

$\Rightarrow 4y(2y+9)-17(2y+9)=0$

$\Rightarrow (2y+9)(4y-17)=0$

$\Rightarrow y=-\frac{9}{2},\frac{17}{4}$

For $y=-\frac{9}{2}$,

$$x+\frac{1}{x}=-\frac{9}{2}$$
$$\frac{x^2+1}{x}=-\frac{9}{2}$$
$$2x^2+9x+2=0$$
$$x=\frac{-9\pm\sqrt{81-16}}{4}=\frac{-9\pm\sqrt{65}}{4}$$

For $y=\frac{17}{4}$,

$$x+\frac{1}{x}=\frac{17}{4}$$
$$\frac{x^2+1}{x}=\frac{17}{4}$$
$$4x^2-17x+4=0$$
$$4x^2-16x-x+4=0$$
$$(4x-1)(x-4)=0$$
$$x=\frac{1}{4},4$$

Therefore, the roots are $\frac{1}{4}$, 4, $\frac{-9+\sqrt{65}}{4}$ and $\frac{-9-\sqrt{65}}{4}$.

CASE III: $a\left(x^2+\frac{1}{x^2}\right)+b\left(x-\frac{1}{x}\right)+c=0$

For this type, we use the identity $x^2+\frac{1}{x^2}=\left(x-\frac{1}{x}\right)^2+2$. The equation then becomes $a\left[\left(x-\frac{1}{x}\right)^2+2\right]+b\left(x-\frac{1}{x}\right)$ $+c=0$. Now, put $x-\frac{1}{x}=y$ to obtain the quadratic equation $ay^2+by+(c+2a)=0$ and follow the same procedure as mentioned in case II.

Example: Solve $6\left(x^2+\frac{1}{x^2}\right)-25\left(x-\frac{1}{x}\right)+12=0$.

Solution: The given equation can be rewritten as $6\left[\left(x-\frac{1}{x}\right)^2+2\right]-25\left(x-\frac{1}{x}\right)+12=0$.

Let $x-\frac{1}{x}=y$.

$6(y^2+2)-25y+12=0$

$\Rightarrow 6y^2-25y+24=0$

$\Rightarrow 6y^2-9y-16y+24=0$

$\Rightarrow 3y(2y-3)-8(y-3)=0$

$\Rightarrow (2y-3)(3y-8)=0$

$\Rightarrow y=\frac{3}{2},\frac{8}{3}$

For $y = \frac{3}{2}$,

$$x - \frac{1}{x} = \frac{3}{2}$$
$$\frac{x^2 - 1}{x} = \frac{3}{2}$$
$$2x^2 - 2 = 3x$$
$$2x^2 - 3x - 2 = 0$$
$$2x^2 - 4x + x - 2 = 0$$
$$(2x+1)(x-2) = 0$$
$$x = -\frac{1}{2}, 2$$

For $y = \frac{8}{3}$,

$$x - \frac{1}{x} = \frac{8}{3}$$
$$\frac{x^2 - 1}{x} = \frac{8}{3}$$
$$3x^2 - 3 = 8x$$
$$3x^2 - 8x - 3 = 0$$
$$3x^2 - 9x + x - 3 = 0$$
$$(3x+1)(x-3) = 0$$
$$x = -\frac{1}{3}, 3$$

Therefore, the roots are $-\frac{1}{3}, -\frac{1}{2}$, 2 and 3.

CASE IV: $x^{2a} + x^a + b = 0$ and $x^a + x^{-a} = b$

In both the situations, put $x^a = y$ to get a quadratic equation in *y*, that is, $y^2 + y + b = 0$ and $y + \frac{1}{y} = b$, respectively.

Example: Solve $3^{x+2} + 3^{-x} = 10$.

Solution: The given equation can be rewritten as $3^2(3^x) + \frac{1}{3^x} = 10$. Let $3^x = y$.

$$3^2(3^x) + \frac{1}{3^x} = 10$$
$$\Rightarrow 9y + \frac{1}{y} = 10$$
$$\Rightarrow 9y^2 - 10y + 1 = 0$$
$$\Rightarrow 9y^2 - 9y - y + 1 = 0$$
$$\Rightarrow (9y-1)(y-1) = 0$$
$$\Rightarrow y = \frac{1}{9}, 1$$

For $y=\frac{1}{9}$, $3^x=3^{-2}\Rightarrow x=-2$.

For $y=1$, $3^x=3^0\Rightarrow x=0$.

Therefore, the roots are 0 and –2.

CASE V: $(x+a)(x+b)(x+c)(x+d)+k=0$

In this case, club the linear pairs where the sum of the two constants is equal to the sum of the other two constants and then simplify. Let us consider an example for better clarity.

Example: Solve $(x+1)(x+3)(x+4)(x+6)=280$.

Solution: Since $1+6=7=3+4$, we can rewrite the given equation as

$$[(x+1)(x+6)][(x+3)(x+4)]=280\Rightarrow(x^2+7x+6)(x^2+7x+12)=280$$

Let $x^2+7x=y$.

$$(x^2+7x+6)(x^2+7x+12)=280$$
$$\Rightarrow(y+6)(y+12)=280$$
$$\Rightarrow y^2+18y-208=0$$
$$\Rightarrow y^2+26y-8y-208=0$$
$$\Rightarrow(y+26)(y-8)=0$$
$$\Rightarrow y=-26,8$$

For $y=-26$,

$$x^2+7x=-26$$
$$x^2+7x+26=0$$
$$x=\frac{-7\pm\sqrt{49-104}}{2}=\frac{-7\pm\sqrt{-55}}{2}$$

Since $\sqrt{-55}$ is not a real number, there will be no real values of *x* for $y=-26$.

For $y=8$,

$$x^2+7x=8$$
$$x^2+7x-8=0$$
$$x^2+8x-x-8=0$$
$$(x-1)(x+8)=0$$
$$x=1,-8$$

Therefore, the roots are 1 and –8.

MISCONCEPTION

$x^2-5x=0$ is not a quadratic equation as the constant term is 0.

FACT: $x^2-5x=0$ is a quadratic equation as $a=1\neq0$. If either $b=0$ or $c=0$ but not both, then the equation qualifies as a quadratic equation.

EXAMPLES

1. Find the roots of $2x^2 - x - 3 = 0$ by factorization method.

Solution: $2x^2 - x - 3 = 0$

$$\Rightarrow 2x^2 - 3x + 2x - 3 = 0$$
$$\Rightarrow x(2x-3) + 1(2x-3) = 0$$
$$\Rightarrow (2x-3)(x+1) = 0$$
$$\Rightarrow x = \frac{3}{2}, -1$$

2. Solve $8\left(x - \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) = 121.$

Solution: Recall the identity $\left(x - \frac{1}{x}\right)^2 = \left(x + \frac{1}{x}\right)^2 - 4$. Let $x + \frac{1}{x} = y$. Then $\left(x - \frac{1}{x}\right)^2 = y^2 - 4$.

$$8\left(x - \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) = 121$$
$$\Rightarrow 8(y^2 - 4) + 2y - 121 = 0$$
$$\Rightarrow 8y^2 + 2y - 153 = 0$$
$$\Rightarrow 8y^2 + 36y - 34y - 153 = 0$$
$$\Rightarrow 4y(2y+9) - 17(2y+9) = 0$$
$$\Rightarrow (2y+9)(4y-17) = 0$$
$$\Rightarrow y = -\frac{9}{2}, \frac{17}{4}$$

For $y = -\frac{9}{2}$,

$$x + \frac{1}{x} = -\frac{9}{2}$$
$$\frac{x^2+1}{x} = -\frac{9}{2}$$
$$2x^2 + 9x + 2 = 0$$
$$x = \frac{-9 \pm \sqrt{81-16}}{4} = \frac{-9 \pm \sqrt{65}}{4}$$

For $y = \frac{17}{4}$,

$$x + \frac{1}{x} = \frac{17}{4}$$
$$\frac{x^2+1}{x} = \frac{17}{4}$$
$$4x^2 - 17x + 4 = 0$$
$$4x^2 - 16x - x + 4 = 0$$

$$(4x-1)(x-4)=0$$

$$x=\frac{1}{4},4$$

Therefore, the roots are $\frac{1}{4}, 4, \frac{-9+\sqrt{65}}{4}$, and $\frac{-9-\sqrt{65}}{4}$.

3. Solve $2x^{\frac{2}{3}}-x^{\frac{1}{3}}=28$.

Solution: Let $x^{\frac{1}{3}}=y$.

$$2x^{\frac{2}{3}}-x^{\frac{1}{3}}=28$$
$$\Rightarrow 2y^2-y-28=0$$
$$\Rightarrow 2y^2-8y+7y-28=0$$
$$\Rightarrow (y-4)(2y+7)=0$$
$$\Rightarrow y=4,-\frac{7}{2}$$

For $y=4$, $x^{\frac{1}{3}}=4 \Rightarrow x=4^3=64$.

For $y=-\frac{7}{2}$, $x^{\frac{1}{3}}=-\frac{7}{2} \Rightarrow x=-\frac{7^3}{2^3}=-\frac{343}{8}$.

Therefore, the roots are 64 and $-\frac{343}{8}$.

4. Solve $\frac{x}{x+1}+\frac{x+1}{x}=\frac{34}{15}$.

Solution: Let $\frac{x}{x+1}=y$.

$$\frac{x}{x+1}+\frac{x+1}{x}=\frac{34}{15}$$
$$\Rightarrow y+\frac{1}{y}=\frac{34}{15}$$
$$\Rightarrow \frac{y^2+1}{y}=\frac{34}{15}$$
$$\Rightarrow 15y^2-34y+15=0$$
$$\Rightarrow 15y^2-9y-25y+15=0$$
$$\Rightarrow (3y-5)(5y-3)=0$$
$$\Rightarrow y=\frac{5}{3},\frac{3}{5}$$

For $y=\frac{5}{3}$,

$$\frac{x}{x+1}=\frac{5}{3}$$
$$3x=5x+5$$

$$x = -\frac{5}{2}$$

For $y = \frac{3}{5}$,

$$\frac{x}{x+1} = \frac{3}{5}$$
$$5x = 3x + 3$$
$$x = \frac{3}{2}$$

Therefore, the roots are $-\frac{5}{2}$ and $\frac{3}{2}$.

5. Solve $4\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) = 60$.

Solution: The given equation can be rewritten as $4\left[\left(x + \frac{1}{x}\right)^2 - 2\right] - \left(x + \frac{1}{x}\right) = 60$.

Let $x + \frac{1}{x} = y$.

$$4\left[\left(x + \frac{1}{x}\right)^2 - 2\right] - \left(x + \frac{1}{x}\right) = 60$$
$$\Rightarrow 4(y^2 - 2) - y = 60$$
$$\Rightarrow 4y^2 - y - 68 = 0$$
$$\Rightarrow 4y^2 - 17y + 16y - 68 = 0$$
$$\Rightarrow (y+4)(4y-17) = 0$$
$$\Rightarrow y = -4, \frac{17}{4}$$

For $y = -4$,

$$x + \frac{1}{x} = -4$$
$$\frac{x^2+1}{x} = -4$$
$$x^2 + 4x + 1 = 0$$
$$x = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

For $y = \frac{17}{4}$,

$$x + \frac{1}{x} = \frac{17}{4}$$
$$\frac{x^2+1}{x} = \frac{17}{4}$$
$$4x^2 - 17x + 4 = 0$$
$$4x^2 - 16x - x + 4 = 0$$

$$(4x-1)(x-4)=0$$

$$x=\frac{1}{4},4$$

Therefore, the roots are $-2-\sqrt{3}$, $-2+\sqrt{3}$, $\frac{1}{4}$, and 4.

6. Find the roots of the following quadratic equations by factorization method.

(i) $x^2-3x-10=10$ **(ii)** $\sqrt{2}x^2+7x+5\sqrt{2}=0$

Solution: (i) $x^2-3x-10=10 \Rightarrow x^2-3x-20=0$

Observe that -3 cannot be expressed as a sum or difference of two numbers such that their product is -20. Therefore, the factorization method fails here.

(ii) $\sqrt{2}x^2+7x+5\sqrt{2}=0$

$$\Rightarrow \sqrt{2}x^2+2x+5x+5\sqrt{2}=0$$

$$\Rightarrow \sqrt{2}x\left(x+\sqrt{2}\right)+5\left(x+\sqrt{2}\right)=0$$

$$\Rightarrow \left(x+\sqrt{2}\right)\left(\sqrt{2}x+5\right)=0$$

$$\Rightarrow x=-\sqrt{2},-\frac{5}{\sqrt{2}}$$

7. Solve the following quadratic equations:

(i) $9x^2+15x+10=0$ **(ii)** $4x^2-20x+25=0$ **(iii)** $4x^2-7x-15=0$

Solution: (i) On comparing $9x^2+15x+10=0$ with the standard form of quadratic equation, we get $a=9$, $b=15$, and $c=10$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-15\pm\sqrt{15^2-4(9)(10)}}{2(9)}$$

$$=\frac{-15\pm\sqrt{225-360}}{18}$$

$$=\frac{-15\pm\sqrt{-135}}{18}$$

$$=\frac{-5\pm\sqrt{-15}}{6}$$

Therefore, the roots are $\frac{-5-\sqrt{-15}}{6}$ and $\frac{-5+\sqrt{-15}}{6}$.

(ii) $4x^2-20x+25=0$

$$\Rightarrow (2x)^2-2(2x)(5)+(5)^2=0$$

$$\Rightarrow (2x-5)^2=0$$

$$\Rightarrow x=\frac{5}{2},\frac{5}{2}$$

Therefore, the roots are $\frac{5}{2}$ and $\frac{5}{2}$.

(iii) $4x^2 - 7x - 15 = 0$

$\Rightarrow 4x^2 - 12x + 5x - 15 = 0$

$\Rightarrow 4x(x-3) + 5(x-3) = 0$

$\Rightarrow (x-3)(4x+5) = 0$

$\Rightarrow x = 3, -\frac{5}{4}$

Therefore, the roots are $-\frac{5}{4}$ and 3.

RECALL

1. Quadratic equation is a polynomial equation of second degree which in general form is written as $ax^2 + bx + c = 0$, where $a \neq 0$.
2. A quadratic equation can be solved using the factorization method, the square root method, or the quadratic formula method.
3. For an equation $ax^2 + bx + c = 0$, $a \neq 0$, the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ helps determine its two roots.
4. Sum of roots of any quadratic equation $= \frac{-\text{coefficient of } x}{\text{coefficient of } x^2} = \frac{-b}{a}$
5. Product of roots of any quadratic equation $= \frac{\text{constant coefficient}}{\text{coefficient of } x^2} = \frac{c}{a}$
6. The sign of roots of a quadratic equation can be determined by the signs of the sum and the product of the roots.
7. Equations that are not quadratic in nature but can be reduced to a quadratic form by suitable transformations can be solved by the methods for solving quadratic equations.

DESCRIPTIVE QUESTIONS

I. VERY SHORT ANSWER QUESTIONS

1. What is a quadratic expression? Give one example.
2. What is a quadratic equation? Give one example.
3. What is the degree of a quadratic equation?
4. In the equation $ax^2 + bx + c = 0$, what do the letters a, b, and c denote?
5. Name the methods of solving quadratic equations.
6. Solve for x: $x^2 + 5x = 0$
7. Solve the quadratic equation $x^2 = 3x$.
8. Solve: $x^2 = 4$
9. Solve the quadratic equation $7x^2 = 8 - 10x$ by a method of your choice.
10. Solve for x: $3(x^2 - 4) = 5x$
11. Give the formula to find the roots of a quadratic equation.
12. Find the roots of $3x^2 + 4x - 7 = 0$.

13. Find the sum of the roots of $x^2 - 7x - 30 = 0$.

14. Find the product of the roots of $2x^2 + 7x - 9 = 0$.

15. What can you say about the sign of roots if the sum as well as the product of roots is positive?

II. SHORT ANSWER QUESTIONS

16. Solve the quadratic equation $2x^2 + 0.3x - 0.35 = 0$.

17. What are the steps followed to solve a quadratic equation by the method of factorization?

18. Find the roots of $2x^2 - x - 3 = 0$ by factorization method

19. Explain the steps followed to solve a quadratic equation by the square root method.

20. Solve for x: $2x^2 + 7x + 3 = 0$

21. Solve: $2x^2 + x - 4 = 0$

III. LONG ANSWER QUESTIONS

22. Solve $5x^2 + 12x + 6 = 0$ for x. Also, find the sum and the product of roots.

23. Solve: $(x+1)(x+5)(x+2)(x-2) = 13$

24. How is the formula derived to find the solutions of a quadratic equation?

25. Solve: $\frac{x}{x+1} + \frac{x+1}{x} = \frac{10}{3}$

26. Solve: $5\left(x^2 + \frac{1}{x^2}\right) - 8\left(x + \frac{1}{x}\right) = 75$

27. Find the roots of the following quadratic equations by factorization method.

(i) $3x^2 - 7x - 6 = 0$

(ii) $\sqrt{3}x^2 - x - 4\sqrt{3} = 0$

(iii) $2x^2 - x + \frac{1}{8} = 0$

IV. FILL IN THE BLANKS

28. A polynomial of degree _______ is termed as a quadratic polynomial or a quadratic expression.

29. An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ and $a, b, c \in R$, is called a _______ equation.

30. The values of x for which the equation $ax^2 + bx + c = 0$ is satisfied are called the _______ of the quadratic equation.

31. A quadratic equation can have two roots which are either _______ or _______.

32. The _______ method of solving quadratic equation involves factoring the linear coefficient.

33. The roots of the equation $ax^2 + bx + c = 0$ are given by _______.

34. For the quadratic equation $ax^2 + bx + c = 0$, the sum of roots is _______.

35. Product of roots of a quadratic equation is _______.

36. If sum of roots is positive and product of roots is negative, the roots are of opposite sign and the numerically _______ root is positive.

V. TRUE OR FALSE

37. In a quadratic equation $ax^2 + bx + c = 0$, $a \geq 0$ and $b \neq 0, c \neq 0$.

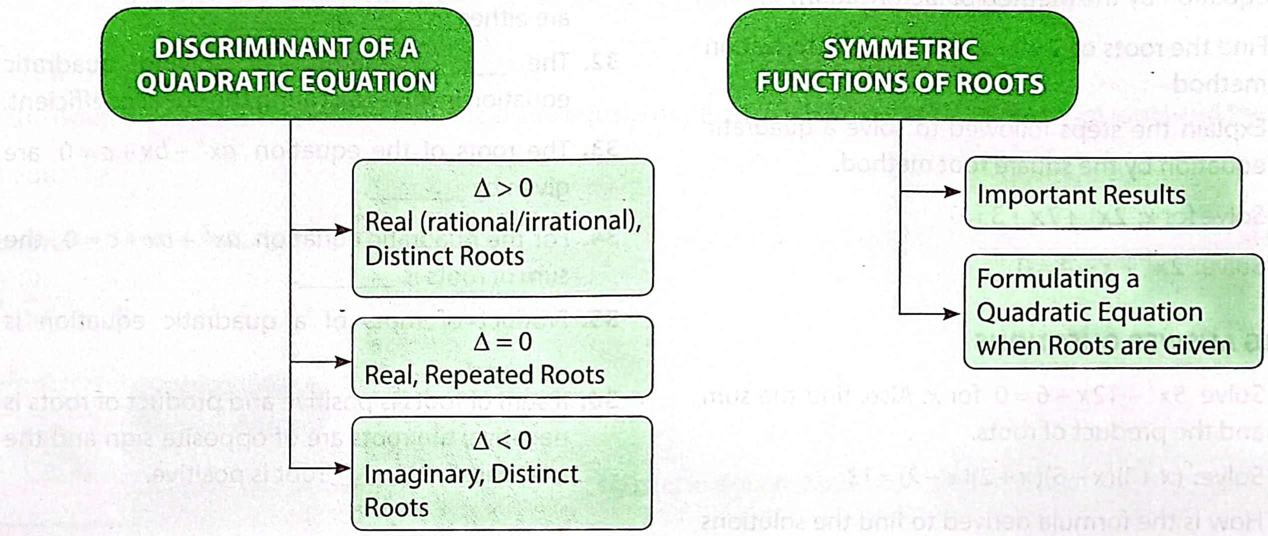
38. The roots of the equation $x^2 + 10x + 25 = 0$ are 5 and −5.

39. The product of roots of $4x^2 + 3x + 5 = 0$ is $\frac{-5}{4}$.

40. The sum of roots of $3x^2 - 4x + 10 = 0$ is $\frac{4}{3}$.

3.04 Discriminant and Symmetric Functions of Roots

MIND MAP



DISCRIMINANT OF A QUADRATIC EQUATION

The sign of the roots of a quadratic equation can be determined by observing the signs of the sum and the product of roots, but to determine the nature, that is, whether the roots are real or imaginary, the calculation of discriminant is a requirement.

The discriminant of a quadratic equation $ax^2+bx+c=0$, $a\neq 0$, is the value b^2-4ac. It is denoted by Δ or D.

$$\Delta = b^2-4ac$$

For instance, the discriminant of the equation $5x^2+x-7=0$ is

$$\begin{aligned}\Delta &= b^2-4ac\\ &= 1^2-4(5)(-7)\\ &= 1+140\\ &= 141\end{aligned}$$

Nature of Roots

Based on the value of the discriminant, the following three cases arise:

Case I: $\Delta > 0$

If the discriminant is positive, then the roots are real and distinct. The roots can further be classified as

(i) rational, if Δ is a perfect square, and

(ii) irrational, if Δ is not a perfect square.

Example: $x^2 - x - 6 = 0$ has real and distinct roots as $\Delta = (-1)^2 - 4(1)(-6) = 25 > 0$. Since Δ is a perfect square, the roots can be further classified as rational.

Key point

In case the roots are irrational, they are conjugates of one another. For instance, if one root is $5+\sqrt{3}$, then the second root will be $5-\sqrt{3}$.

Case II: $\Delta = 0$

If the discriminant is 0, then the roots are real and equal. They are called double roots or repeated roots and their value is equal to $\frac{-b}{2a}$.

Example: For the equation $x^2 - 10x + 25 = 0$, $\Delta = (-10)^2 - 4(25) = 0$. So, the roots are real and equal, and their value is $\frac{-b}{2a} = \frac{-(-10)}{2} = 5$.

Key point

In case $\Delta = 0$, $ax^2 + bx + c$ is a perfect square. In the previous example, $x^2 - 10x + 25 = (x-5)^2$.

Case III: $\Delta < 0$

If the discriminant is negative, then the roots are imaginary and distinct.

Example: For the equation $x^2 + x + 3 = 0$, $\Delta = (1)^2 - 4(3) = -11 < 0$. So, the roots are imaginary and distinct.

Key point

$\sqrt{-1}$ is the imaginary number i, called **iota**.

$$i = \sqrt{-1}$$

Example: $\sqrt{-11}$ can be written as $\sqrt{-1} \times \sqrt{11} = i\sqrt{11}$.

Example: For what value of p will the equation $px^2 + 8x - 2 = 0$ have real roots?

Solution: Any quadratic equation has real roots if $\Delta \geq 0$. So, for $px^2 + 8x - 2 = 0$, the roots will be real when

$$b^2 - 4ac \geq 0$$
$$(8)^2 - 4(p)(-2) \geq 0$$
$$64 + 8p \geq 0$$
$$8p \geq -64$$
$$p \geq -8$$

Example: Determine the value(s) of k for which the equation $4x^2 - 3kx + 9 = 0$ has equal roots?

Solution: Any quadratic equation has equal roots if $\Delta = 0$. So, for $4x^2 - 3kx + 9 = 0$, the roots will be equal when

$$b^2 - 4ac = 0$$
$$(-3k)^2 - 4(4)(9) = 0$$
$$9k^2 - 144 = 0$$
$$k^2 = 16$$
$$k = \pm 4$$

SYMMETRIC FUNCTIONS OF ROOTS

Symmetric functions are those expressions that do not change in value on interchanging the variables in it. For instance, the value of $x + y$ remains same if x and y are interchanged, that is, $y + x = x + y$. Here, $x + y$ is called a symmetric function of x and y.

Similarly, if two roots of a quadratic equation are x_1 and x_2, then the following functions are symmetric:

(i) $x_1 x_2$ **(ii)** $x_1 + x_2$

(iii) $\frac{1}{x_1^2} + \frac{1}{x_2^2}$ **(iv)** $\frac{x_1^2}{x_2} + \frac{x_2^2}{x_1}$

It is also observed that the following functions are not symmetric:

(i) $x_1 - x_2$

(ii) $x_1^2 - x_2^2$

Key points

- $x_1 + x_2$ and $x_1 x_2$ are called elementary symmetric functions of the roots x_1 and x_2.
- $|x_1 - x_2|$ is also a symmetric function of the roots as $|x_1 - x_2| = |x_2 - x_1|$.

Some Important Results

For two roots α and β, the following results hold true.

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(ii) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

(iii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

(iv) $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$

(v) $\alpha^4 + \beta^4 = (\alpha + \beta)^4 - 4\alpha\beta(\alpha^2 + \beta^2) - 6(\alpha\beta)^2$

(vi) $\alpha^4 - \beta^4 = (\alpha - \beta)(\alpha + \beta)\left[(\alpha + \beta)^2 - 2\alpha\beta\right]$

(vii) $\alpha^5 + \beta^5 = (\alpha + \beta)^5 - 5\alpha\beta(\alpha^3 + \beta^3) - 10(\alpha\beta)^2(\alpha + \beta)$

(viii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha + \beta)^2}{\alpha\beta} - 2$

(ix) $|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

MISCONCEPTION

If a quadratic equation has irrational roots, then they do not occur in conjugate pairs.

FACT: If a quadratic equation has irrational roots, then they always occur in conjugate pairs, that is, if one root is $a+\sqrt{b}$, then the other root will be $a-\sqrt{b}$.

Example: In a quadratic equation $x^2-10x-11=0$, one root is $5-\sqrt{3}$ and the other is $5+\sqrt{3}$.

Formulating a Quadratic Equation when Roots are Given

A quadratic equation whose roots are α and β is of the form $x^2-(\alpha+\beta)x+\alpha\beta=0$, that is, x^2 – (sum of roots) x + (product of roots) = 0.

Example: Find the quadratic equation whose roots are $\frac{-2}{3}$ and $\frac{1}{4}$.

Solution: Let $\alpha=\frac{-2}{3}$ and $\beta=\frac{1}{4}$. Then the equation is given by $x^2-(\alpha+\beta)x+\alpha\beta=0$.

$$\alpha+\beta=\frac{-2}{3}+\frac{1}{4}=\frac{-5}{12}$$

$$\alpha\beta=\frac{-2}{3}\times\frac{1}{4}=\frac{-1}{6}$$

$$\therefore \quad x^2-(\alpha+\beta)x+\alpha\beta=0 \Rightarrow x^2+\frac{5}{12}x-\frac{1}{6}=0 \Rightarrow 12x^2+5x-2=0$$

Example: Find the quadratic equation whose one root is $3+\sqrt{5}$.

Solution: If one root of a quadratic equation is $3+\sqrt{5}$, then the other root is $3-\sqrt{5}$ as irrational roots always occur in pairs.

$\therefore$ Sum of roots $=3+\sqrt{5}+3-\sqrt{5}=6$

Product of roots $=(3+\sqrt{5})(3-\sqrt{5})=9-5=4$

Hence, the quadratic equation is x^2 – (sum of roots) x + (product of roots) = 0 $\Rightarrow x^2-6x+4=0$.

A new quadratic equation can always be formed by applying specific operations on the roots of a given quadratic equation. If α and β are the roots of $ax^2+bx+c=0$, then let us see what the roots and the quadratic equations are in the following scenarios:

If the Roots are	New Roots	New Equation
p more than α and β	$\alpha+p,\ \beta+p$	$a(x-p)^2+b(x-p)+c=0$
p less than α and β	$\alpha-p,\ \beta-p$	$a(x+p)^2+b(x+p)+c=0$
p times α and β	$p\alpha,\ p\beta$	$a\left(\frac{x}{p}\right)^2+b\left(\frac{x}{p}\right)+c=0$
$\frac{1}{p}$ times α and β	$\frac{\alpha}{p},\ \frac{\beta}{p}$	$a(px)^2+b(px)+c=0$

reciprocals of α and β	$\frac{1}{\alpha}, \frac{1}{\beta}$	$a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c = 0$ $\Rightarrow cx^2 + bx + a = 0$
negatives of α and β	$-\alpha$ and $-\beta$	$a(-x)^2 + b(-x) + c = 0$ $\Rightarrow ax^2 - bx + c = 0$
squares of α and β	α^2 and β^2	$a\left(\sqrt{x}\right)^2 + b\left(\sqrt{x}\right) + c = 0$ $\Rightarrow ax + b\sqrt{x} + c = 0$
cubes of α and β	α^3 and β^3	$a\left(\sqrt[3]{x}\right)^2 + b\left(\sqrt[3]{x}\right) + c = 0$ $\Rightarrow ax^{\frac{2}{3}} + bx^{\frac{1}{3}} + c = 0$

MISCONCEPTION

If $\Delta = 0$, the roots of the quadratic equation $ax^2 + bx + c = 0$ are always zero.

FACT: If $\Delta = 0$, the quadratic expression $ax^2 + bx + c$ is a perfect square and the equation has repeated roots with value equal to $\frac{-b}{2a}$.

EXAMPLES

1. If α and β are the roots of $6x^2 + 7x - 10 = 0$, then find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

Solution: Since α and β are the roots of $6x^2 + 7x - 10 = 0$, we get

$$\alpha + \beta = -\frac{7}{6}$$

$$\alpha\beta = \frac{-10}{6} = \frac{-5}{3}$$

$$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-\frac{7}{6}}{\frac{-5}{3}} = \frac{7}{10}$$

2. Determine the nature of the roots of the following equations:

(i) $15x^2 - 28 = x$ **(ii)** $25x^2 + 30x + 7 = 0$ **(iii)** $2x^2 + 0.3x - 0.35 = 0$

Solution: (i) $15x^2 - 28 = x \Rightarrow 15x^2 - x - 28 = 0$

Here, $a = 15$, $b = -1$, and $c = -28$.

$D = b^2 - 4ac$
$= (-1)^2 - 4(15)(-28)$
$= 1681 > 0$

Therefore, the roots are real and distinct. Since $1681 = (41)^2$, the roots can be further classified as rational.

(ii) $25x^2 + 30x + 7 = 0$

Here, $a = 25$, $b = 30$, and $c = 7$.

$D = b^2 - 4ac$
$= (30)^2 - 4(25)(7)$
$= 900 - 700$
$= 200 > 0$

Therefore, the roots are real and distinct. Since D is not a perfect square, the roots can be further classified as irrational.

(iii) $2x^2 + 0.3x - 0.35 = 0$

Here, $a = 2$, $b = 0.3$, and $c = -0.35$.

$D = b^2 - 4ac$
$= (0.3)^2 - 4(2)(-0.35)$
$= 0.09 + 2.8$
$= 2.89 > 0$

Therefore, the roots are real and distinct. Since $2.89 = (1.7)^2$, the roots can be further classified as rational.

3. Find the quadratic equation whose one root is $1-\sqrt{3}$.

Solution: Since irrational roots always occur in pairs, the second root becomes $1+\sqrt{3}$.

$\therefore$ Sum of roots $= 1-\sqrt{3}+1+\sqrt{3} = 2$

Product of roots $= \left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right) = 1-3 = -2$

Hence, the quadratic equation is x^2 – (sum of roots) x + (product of roots) = 0 $\Rightarrow x^2 - 2x - 2 = 0$.

4. Find the quadratic equation whose roots are $\frac{-3}{5}$ and $\frac{4}{5}$?

Solution: Let $\alpha = \frac{-3}{5}$ and $\beta = \frac{4}{5}$. Then the equation is given by $x^2 - (\alpha+\beta)x + \alpha\beta = 0$.

$$\alpha + \beta = \frac{-3}{5} + \frac{4}{5} = \frac{1}{5}$$

$$\alpha\beta = \frac{-3}{5} \times \frac{4}{5} = \frac{-12}{25}$$

$$\therefore \quad x^2 - (\alpha+\beta)x + \alpha\beta = 0 \Rightarrow x^2 - \frac{1}{5}x - \frac{12}{25} = 0 \Rightarrow 25x^2 - 5x - 12 = 0$$

5. Find the quadratic equation whose one root is $7+\sqrt{11}$.

Solution: Since irrational roots always occur in pairs, the second root becomes $7-\sqrt{11}$.

$\therefore$ Sum of roots $= 7+\sqrt{11}+7-\sqrt{11} = 14$

Product of roots $= \left(7+\sqrt{11}\right)\left(7-\sqrt{11}\right) = 49-11 = 38$

Hence, the quadratic equation is x^2 – (sum of roots) x + (product of roots) = 0 $\Rightarrow x^2 - 14x + 38 = 0$.

6. If $a \neq b$, then what can you say about the roots of $2(a^2+b^2)x^2+2(a+b)x+1=0$?

Solution: Here, $a'=2(a^2+b^2)$, $b'=2(a+b)$, and $c'=1$.

$$D = b'^2 - 4a'c'$$
$$= 4(a+b)^2 - 8(a^2+b^2)(1)$$
$$= 4a^2+4b^2+8ab-8a^2-8b^2$$
$$= 8ab-4a^2-4b^2$$
$$= -4(a^2+b^2-2ab)$$
$$= -4(a-b)^2$$

Since $a \neq b$, $(a-b)^2$ will always give a positive value and so, $-4(a-b)^2 < 0$. Therefore, the roots are imaginary and distinct.

7. If α and β are the roots of $x^2+x+1=0$, then find the value of $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$.

Solution: Since α and β are the roots of $x^2+x+1=0$, we get

$$\alpha+\beta = -\frac{1}{1} = -1$$

$$\alpha\beta = \frac{1}{1} = 1$$

$$\therefore \quad \frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{(\alpha+\beta)^2}{\alpha\beta} - 2 = \frac{(-1)^2}{1} - 2 = -1$$

RECALL

1. The discriminant, denoted by Δ or D, of a quadratic equation $ax^2+bx+c=0$, $a \neq 0$, is the value b^2-4ac.
2. For $\Delta > 0$, the roots are real (rational when Δ is a perfect square, irrational when Δ is not a perfect square) and distinct.
3. For $\Delta = 0$, the roots are real and repeated with value equal to $\frac{-b}{2a}$.
4. For $\Delta < 0$, the roots are imaginary and distinct.
5. A symmetric function of roots α and β is an expression whose value remains unchanged when α and β are interchanged.
6. $\alpha+\beta$ and $\alpha\beta$ are called elementary symmetric functions of the roots α and β.
7. A quadratic equation whose roots are α and β is of the form $x^2-(\alpha+\beta)x+\alpha\beta=0$, that is, x^2 – (sum of roots) x + (product of roots) = 0.
8. A new quadratic equation can always be formed by applying specific operations on the roots of a given quadratic equation.

DESCRIPTIVE QUESTIONS

I. VERY SHORT ANSWER QUESTIONS

1. Define discriminant.
2. What does the discriminant of a quadratic equation tell us about?
3. If $\Delta < 0$, then comment on the nature of roots.
4. If $\Delta = 0$, then comment on the nature of roots.
5. If one root of a quadratic equation is $\sqrt{m} + \sqrt{n}$, then what can you infer about the second root?
6. Without solving, examine the nature of roots of the following equations:

   (i) $2x^2 + 2x + 3 = 0$ (ii) $2x^2 - 7x + 3 = 0$
7. Comment on the nature of roots of the following equations:

   (i) $x^2 - 5x - 2 = 0$ (ii) $4x^2 - 4x + 1 = 0$
8. What do you understand by symmetric functions of roots?
9. If a quadratic equation has roots which are reciprocals of α and β, then write down its general form.
10. If a quadratic equation has roots which are p less than α and β, then write down its roots and general form.

II. SHORT ANSWER QUESTIONS

11. If α and β are the roots of $x^2 + x + 2 = 0$, then find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.
12. Discuss about the nature of roots when $\Delta > 0$.
13. For what value of m will the equation $9x^2 + 11x - m = 0$ have imaginary roots?
14. Find the discriminant of the equation $x^2 - x - 6 = 0$. Are the roots equal?
15. Determine the value(s) of t for which the equation $7x^2 - 8tx + 3 = 0$ has equal roots?
16. If α and β are the roots of the equation $px^2 + qx + r = 0$, then what is the value of $\alpha^3\beta + \beta^3\alpha$?
17. Find the quadratic equation whose one root is $10 - \sqrt{5}$.
18. If $\alpha + \beta = 5$ and $\alpha\beta = \frac{1}{5}$, then find the value of $|\alpha - \beta|$.
19. If the roots of an equation are 9 more than the roots of $x^2 - 2x - 8 = 0$, then find the equation.
20. If the roots of an equation are cubes of the roots of $x^2 - 5x + 6 = 0$, then find the equation.

III. LONG ANSWER QUESTIONS

21. If -5 is one of the roots of $2x^2 + 2px - 15 = 0$ and the equation $p(x^2 + x) + k = 0$ has equal roots, then find the value of k.
22. Find the discriminant and comment on the nature of the roots of the following equations:

   (i) $x^2 - \sqrt{6}x + 9 = 0$ (ii) $3x^2 - 5x - 1 = 0$

   (iii) $13x^2 + x + 100 = 0$ (iv) $x^2 - 12x + 36 = 0$
23. If α, β are the roots of the equations $x^2 - px + q = 0$ and $\alpha > 0, \beta > 0$, then find the values of

   (i) $\alpha^5 + \beta^5$ (ii) $\alpha^{\frac{1}{4}} + \beta^{\frac{1}{4}}$
24. If α, β are the roots of $x^2 - 10x + 25$, then find the values of

   (i) $\alpha^3 + \beta^3$ (ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

IV. FILL IN THE BLANKS

25. The roots of a quadratic equation are real and distinct if ________.
26. For ________, the roots of a quadratic equation are real and equal.
27. $\alpha + \beta$ and $\alpha\beta$ are called ________ symmetric functions of the roots α and β.
28. The roots are rational if a, b, c are rational and D is a ________ ________.
29. If the roots of a quadratic equation are ________ times the roots of $ax^2 + bx + c = 0$, then the equation is $a(px)^2 + b(px) + c = 0$.
30. If one root of a quadratic equation is $a + \sqrt{b}$, then the other root is ________.

V. TRUE OR FALSE

31. If $\Delta = 0$, the quadratic expression $ax^2 + bx + c$ is a perfect square.

32. If $\Delta < 0$ and $a, b, c \in R$, then the roots are complex and distinct.

33. The formula of discriminant is $\Delta = b^2 + 4ac$.

34. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

35. A quadratic equation whose roots are α and β is of the form $x^2 - (\alpha + \beta)x - \alpha\beta = 0$.

36. $\alpha^3 + \beta^3 = (\alpha + \beta)^3 + 3\alpha\beta(\alpha + \beta)$

37. A symmetric function is an expression whose value changes when the variables are interchanged.

38. $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3 + \beta^3}{\alpha\beta}$

3.05 Remainder and Factor Theorem

M I N D M A P

OPERATIONS ON POLYNOMIALS

Remainder Theorem

Let $p(x)$ be a polynomial of degree greater than or equal to one and let a be any real number. If $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.

Factor Theorem

Let $p(x)$ be a polynomial of degree greater than or equal to one and let a be any real number. Then $x - a$ is a factor of $p(x)$ iff $p(a) = 0$.

INTRODUCTION

Factors of a polynomial are usually determined by the methods of long division, synthetic division, or other methods of factoring. Using the remainder theorem or the factor theorem for this purpose proves useful and even handy. These theorems allow us to evaluate a polynomial at a given number which later helps in determining the remainder of the polynomial after general division or even determine a factor of that polynomial. Hence, we can say that the remainder theorem and the factor theorem provide efficient avenues for testing whether certain numbers are roots of polynomials or not, even though these theorems are somewhat trial-and-error methods.

General Division

When a natural number n is divided by another natural number m, where $m \leq n$, the remainder is either 0 or a natural number $\boldsymbol{r < m}$.

Example: When 27 is divided by 12, it leaves a remainder 3. Here, 27 can be written as $27 = (12 \times 2) + 3$.

$\therefore$ **Dividend = (Divisor × Quotient) + Remainder**

We now extend this method to division of a non-zero polynomial $p(x)$ by another non-zero polynomial $g(x)$, where degree of $g(x)$ $\leq$ degree of $p(x)$. We can find polynomials $q(x)$ and $r(x)$ such that $p(x) = g(x) \times q(x) + r(x)$, where $r(x) = 0$ or degree of $r(x)$ < degree of $g(x)$.

Here, $p(x)$ is the dividend, $g(x)$ is the divisor, $q(x)$ is the quotient, and $r(x)$ is the remainder.

Division of a Polynomial by Another Polynomial

Following are the steps for carrying out the mentioned division:

Step 1: Arrange the terms of the dividend and the divisor in descending order of their degrees.

Step 2: Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

Step 3: Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend.

Step 4: Consider the remainder as the new dividend and follow steps 2 and 3 again.

Step 5: Repeat this process till we obtain a remainder which is either 0 or a polynomial of degree less than that of the divisor.

For instance, let us divide x^2+5x-6 by $x+5$.

$$\begin{array}{r|l} & \quad x \\ x+5 & x^2+5x-6 \\ & x^2+5x \\ & (-)(-) \\ \hline & \qquad -6 \end{array}$$

Since degree of -6 < degree of $x+5$, we stop here. So, x^2+5x-6 can be written as $x(x+5)-6$.

REMAINDER THEOREM

Let $p(x)$ be any polynomial of degree greater than or equal to 1 and let a be any real number. If $p(x)$ is divided by the linear polynomial $x-a$, then the remainder is $p(a)$.

Proof: Let $p(x)$ be a polynomial of degree greater than or equal to 1 and let a be a real number.

Suppose that when $p(x)$ is divided by $x-a$, the quotient is $q(x)$ and the remainder is $r(x)$, that is, $p(x)=(x-a)q(x)+r(x)$.

Since degree of $x-a$ is 1 and degree of remainder is always less than the degree of divisor, degree of $r(x)$ will be 0. This means that $r(x)$ is a constant, say r.

$\therefore \qquad p(x)=(x-a)q(x)+r$

If $x=a$, then the above equation becomes $p(a)=(a-a)q(a)+r \Rightarrow p(a)=r$ and this proves the theorem.

Following is a table that lists the general form of some divisions and their respective remainders when the dividend is $f(x)$.

Divisor	Remainder
$x-a$	$f(a)$
$x+a$	$f(-a)$
$ax+b$	$f\left(-\dfrac{b}{a}\right)$
$ax-b$	$f\left(\dfrac{b}{a}\right)$
x	$f(x)$

Example: Divide $p(x)=x+3x^2-1$ by $g(x)=1+x$.

Solution: Let us first rewrite $p(x)$ and $g(x)$ as $p(x)=3x^2+x-1$ and $g(x)=x+1$.

$$
\begin{array}{r l}
 & 3x-2 \\
x+1 \overline{)} & 3x^2+x-1 \\
 & 3x^2+3x \\
 & \underline{\scriptstyle(-)(-)} \\
 & -2x-1 \\
 & -2x-2 \\
 & \underline{\scriptstyle(+)\ (+)} \\
 & 1
\end{array}
$$

$\therefore \qquad q(x)=3x-2$ and $r(x)=1$

Hence, $p(x)=g(x)\times q(x)+r(x) \Rightarrow 3x^2+x-1=(x+1)(3x-2)+1$.

Example: Divide the polynomial $3x^4-4x^3-3x-1$ by $x-1$.

Solution: Let $p(x)=3x^4-4x^3-3x-1$ and $g(x)=x-1$.

$$
\begin{array}{r l}
 & 3x^3-x^2-x-4 \\
x-1 \overline{)} & 3x^4-4x^3 \quad -3x-1 \\
 & 3x^4-3x^3 \\
 & \underline{\scriptstyle(-)(+)} \\
 & -x^3-0x^2 \\
 & -x^3+x^2 \\
 & \underline{\scriptstyle(+)\ (-)} \\
 & -x^2-3x \\
 & -x^2+x \\
 & \underline{\scriptstyle(+)\ (-)} \\
 & -4x-1 \\
 & -4x+4 \\
 & \underline{\scriptstyle(+)\ (-)} \\
 & -5
\end{array}
$$

$\therefore \quad 3x^4-4x^3-3x-1=(x-1)(3x^3-x^2-x-4)-5$

Here, the remainder is –5. Let us check if the remainder theorem is satisfied or not.

Put $x=1$ in $p(x)$ and simplify.

$$
\begin{aligned}
p(1) &= 3(1)^4-4(1)^3-3(1)-1 \\
&= 3-4-3-1 \\
&= -5 \text{ which is the remainder}
\end{aligned}
$$

Hence, remainder theorem is satisfied.

Example: Find the remainder obtained on dividing $p(x)=x^3+1$ by $x+1$.

Solution: By the remainder theorem, the remainder will be $p(-1)$.

$$p(-1)=(-1)^3+1=-1+1=0$$

So, without the actual long division, we easily determined the remainder in one single substitution and simplification.

Horner's Method of Synthetic Division

If $f(x)=x^3-3x^2+5x-9$ is divided by $g(x)=x-4$, the remainder obtained is $f(4)=4^3-3(4)^2+5(4)-9=64-48+20-9=84-57=27$.

This remainder can be determined by another method of writing numerical coefficients of $f(x)$ in descending order, as shown in the figure. The divisor 4 is written in the leftmost column and the coefficients of the quadratic equation are written in the first row. Then, the first coefficient is written in the last row, multiplied by 4 and subtracted from the next coefficient. The result is then written in the last row, multiplied by 4 and subtracted from the next coefficient. This process continues till the last value obtained in the last row is the remainder.

4	1	–3	5	–9	
	0	4	4	36	
	1	1	9	27	→ Remainder

This method of division of polynomials is called Horner's Method of Synthetic Division.

FACTOR THEOREM

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let a be any real number. Then, $x-a$ is a factor of $f(x)$ if $f(a)=0$, or $f(a)=0$ if $x-a$ is a factor of $f(x)$. In simpler terms, the factor theorem states that for a polynomial $f(x)$ of degree greater than or equal to 1, $x-a$ is a factor of $f(x)$ iff (read as if and only if) $f(a)=0$, where a is any real number.

Examples:

(i) $(x + a)$ is a factor of a polynomial $p(x)$ if $p(-a) = 0$ and vice versa.

(ii) $(ax - b)$ is a factor of a polynomial $p(x)$ if $p\left(\frac{b}{a}\right)=0$ and vice versa.

(iii) $(ax + b)$ is a factor of a polynomial $p(x)$ if $p\left(-\frac{b}{a}\right)=0$ and vice versa.

(iv) $(x - a)(x - b)$ is a factor of a polynomial $p(x)$ if $p(a) = 0$ and $p(b) = 0$, and vice versa.

Key point

The term *iff* is used to denote forward implication as well as backward implication in one statement. For instance, "if *p* then *q*" and "if *q* then *p*" can be written in one statement as "*p* iff *q*".

How to Use Factor Theorem

The following steps are followed to find the factors of a polynomial using factor theorem:

(i) Check if $f(-c) = 0$. If yes, then $(x + c)$ is a factor of the polynomial $f(x)$.

(ii) Check if $f\left(\frac{d}{c}\right)=0$. If yes, then $(cx - d)$ is a factor of the polynomial $f(x)$.

(iii) Check if $f\left(-\frac{d}{c}\right)=0$. If yes, then $(cx + d)$ is a factor of the polynomial $f(x)$.

(iv) Check if $f(c) = 0$ and $f(d) = 0$. If yes, then $(x - c)(x - d)$ is a factor of the polynomial $f(x)$.

Example: Show that $(x-1)$ is a factor of the polynomial x^3+2x^2+2x-5.

Solution: Let $f(x) = x^3 + 2x^2 + 2x - 5$. Then, by factor theorem, $(x-1)$ is a factor of $f(x)$ if $f(1) = 0$.

$$f(1) = (1)^3 + 2(1)^2 + 2(1) - 5$$
$$= 1 + 2 + 2 - 5$$
$$= 0$$

Hence, it is proved that $(x-1)$ is a factor of the polynomial $x^3 + 2x^2 + 2x - 5$.

Example: Find the value of a if $(x + a)$ is a factor of $x^3 + ax^2 - 4x + a + 5$.

Solution: Since $(x + a)$ is a factor of $f(x) = x^3 + ax^2 - 4x + a + 5$, by factor theorem $f(-a) = 0$.

$$f(-a) = (-a)^3 + a(-a)^2 - 4(-a) + a + 5$$
$$0 = -a^3 + a^3 + 4a + a + 5$$
$$a = 1$$

Hence, the value of a is 1.

MISCONCEPTION

The remainder as well as the factor theorem link the remainder of division by a binomial to the value of the function at a point.

FACT: Basically, the remainder theorem links the remainder of division by a binomial to the value of the function at a point, whereas the factor theorem links the factors of a polynomial to its zeros.

EXAMPLES

1. Without actual division, find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

   Solution: Let $p(x) = x^4 + x^3 - 2x^2 + x + 1$. Then, by remainder theorem, $p(-1)$ will be the remainder.

   $$p(1) = (1)^4 + (1)^3 - 2(1)^2 + (1) + 1$$
   $$= 1 + 1 - 2 + 1 + 1$$
   $$= 2$$

   Hence, the remainder is 2 when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

2. Find the remainder when $4x^3 - 3x^2 + 2x - 4$ is divided by $x + 1$.

   Solution: Let $f(x) = 4x^3 - 3x^2 + 2x - 4$. Then, by remainder theorem, $f(-1)$ will be the remainder.

   $$f(x) = 4(-1)^3 - 3(-1)^2 + 2(-1) - 4$$
   $$= -4 - 3 - 2 - 4$$
   $$= -13$$

   Hence, the remainder is -13 when $4x^3 - 3x^2 + 2x - 4$ is divided by $x + 1$.

3. What is the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by $5 + 2x$.

   Solution: Let $p(x) = x^3 + 3x^2 + 3x + 1$. Then by remainder theorem, $p\left(-\frac{5}{2}\right)$ will be the remainder.

$$p\left(-\frac{5}{2}\right)=\left(-\frac{5}{2}\right)^3+3\left(-\frac{5}{2}\right)^2+3\left(-\frac{5}{2}\right)+1$$
$$=-\frac{125}{8}+\frac{75}{4}-\frac{15}{2}+1$$
$$=-\frac{27}{8}$$

Hence, the remainder is $-\frac{27}{8}$ when x^3+3x^2+3x+1 is divided by $5+2x$.

4. For what value of k is $2x^2+kx+\sqrt{2}$ exactly divisible by $x-1$?

Solution: Let $p(x)=2x^2+kx+\sqrt{2}$. Since $p(x)$ is exactly divisible by $x-1$, $p(1)=0$ by the factor theorem.

$$p(1)=2(1)^2+k(1)+\sqrt{2}$$
$$0=2+k+\sqrt{2}$$
$$k=-\left(2+\sqrt{2}\right)$$

Hence, $2x^2+kx+\sqrt{2}$ is exactly divisible by $x-1$ for $k=-\left(2+\sqrt{2}\right)$.

5. If ax^3+bx^2+x-6 has $x+2$ as a factor and leaves remainder 4 when divided by $x-2$, then find the values of a and b.

Solution: Let $p(x)=ax^3+bx^2+x-6$. Since $x+2$ is a factor of $p(x)$, $p(-2)=0$ by the factor theorem.

$$p(-2)=a(-2)^3+b(-2)^2+(-2)-6$$
$$0=-8a+4b-2-6$$
$$8a=4b-8$$
$$a=\frac{b-2}{2}$$

It is given that $p(x)$ leaves remainder 4 when $x-2$. So, the remainder is $p(2)$ by the remainder theorem.

$$p(2)=a(2)^3+b(2)^2+(2)-6$$
$$4=8a+4b+2-6$$
$$8a+4b=8$$
$$2a+b=2$$
$$b-2+b=2 \qquad \left[\because a=\frac{b-2}{2}\right]$$
$$2b=4$$
$$b=2$$

Therefore, the value of a is $\frac{b-2}{2}=\frac{2-2}{2}=0$ and the value of b is 2.

6. If x^2-1 is a factor of $ax^4+bx^3+cx^2+dx+e$, then show that $a+c+e=b+d=0$.

Solution: Let $p(x)=ax^4+bx^3+cx^2+dx+e$.

Since x^2-1 is a factor of $p(x)$, this means that $x^2-1=(x+1)(x-1)$ is also a factor of $p(x)$.

So, by factor theorem, $p(1)=0$ and $p(-1)=0$.

$$p(1)=a(1)^4+b(1)^3+c(1)^2+d(1)+e$$
$$0=a+b+c+d+e$$
$$p(-1)=a(-1)^4+b(-1)^3+c(-1)^2+d(-1)+e$$
$$0=a-b+c-d+e$$

On adding the two equations, we get $2(a+c+e)=0 \Rightarrow a+c+e=0$ and on subtracting the two equations, we get $2(b+d)=0 \Rightarrow b+d=0$.

Hence, we can conclude that $a+c+e=b+d=0$.

7. If the polynomials $2x^3+ax^2+3x-5$ and x^3+x^2-2x+a leave the same remainder when divided by $x-2$, find the value of *a*. Also, find the remainder in each case.

Solution: Let $f(x)=2x^3+ax^2+3x-5$ and $g(x)=x^3+x^2-2x+a$.

By the remainder theorem, when $f(x)$ and $g(x)$ are divided by $x-2$, the remainder obtained are $f(2)$ and $g(2)$, respectively.

Since both the remainders are same, $f(2)=g(2)$.

$$f(2)=g(2)$$
$$2(2)^3+a(2)^2+3(2)-5=(2)^3+(2)^2-2(2)+a$$
$$16+4a+6-5=8+4-4+a$$
$$3a=-9$$
$$a=-3$$

Hence, the value of a is -3 and the remainder in each case is $17+4(-3)=17-12=5$.

RECALL

1. When a natural number *n* is divided by a natural number *m*, where $m \le n$, then the remainder is either 0 or a natural number $r<m$.
2. Dividend = (Divisor × Quotient) + Remainder
3. **Remainder Theorem:** Let $p(x)$ be any polynomial of degree greater than or equal to 1 and let *a* be any real number. If $p(x)$ is divided by the linear polynomial $x-a$, then the remainder is $p(a)$.
4. Horner's Method of Synthetic Division is another method of determining the remainder of division of a polynomial by a binomial.
5. **Factor Theorem:** For a polynomial $f(x)$ of degree greater than or equal to 1, $x-a$ is a factor of $f(x)$ iff $f(a)=0$, where *a* is any real number.

DESCRIPTIVE QUESTIONS

I. VERY SHORT ANSWER QUESTIONS

1. State remainder theorem.
2. State factor theorem.
3. Find the zero of the polynomial $3x+2$.
4. If 2 is a root of $kx^4-11x^3+kx^2+13x+2$, then what is the value of *k*?
5. Show that $f(x)=x^3+x^2-5x+3$ is divisible by $x+3$.

6. Check if $x+3$ and $x-2$ are factors of x^2+x-6.
7. Using Factor theorem, check if $x+1$ is a factor of x^2-3x-4.

II. SHORT ANSWER QUESTIONS

8. Find the remainder when $x-4$ divides $2x^3-5x^2+6x-12$.
9. Divide x^3-6x^2+9x+3 by $x-2$. Find the quotient and the remainder.
10. Divide $3x^3-6x^2+13x+60$ by $x+2$. Find the quotient and the remainder.
11. Divide $4x^3-12x^2+11x-5$ by $2x-1$.
12. Find the remainder when $2x^3-5x^2+9x-8$ is divided by $x-3$. Use remainder theorem.
13. Find the remainder when $3x^4-6x^2-8x+2$ is divided by $x+2$.
14. What is the remainder when $2x^4-5x^3+3x^2-2x+9$ is divided by $3x-2$?
15. The polynomials $p(x)=3x^2-5x+12$ and $g(x)=4x^2+ax-15$ leave the same remainder when divided by $x-2$. Find *a*.

III. LONG ANSWER QUESTIONS

16. Give the proof of remainder theorem.
17. Find the remainder when $f(x)=x^3+x^2+x+6$ is divided by $x-2$ using long division method. Verify the remainder by using a suitable theorem.
18. Find all the factors of $x^3+13x^2+32x+20$.

IV. FILL IN THE BLANKS

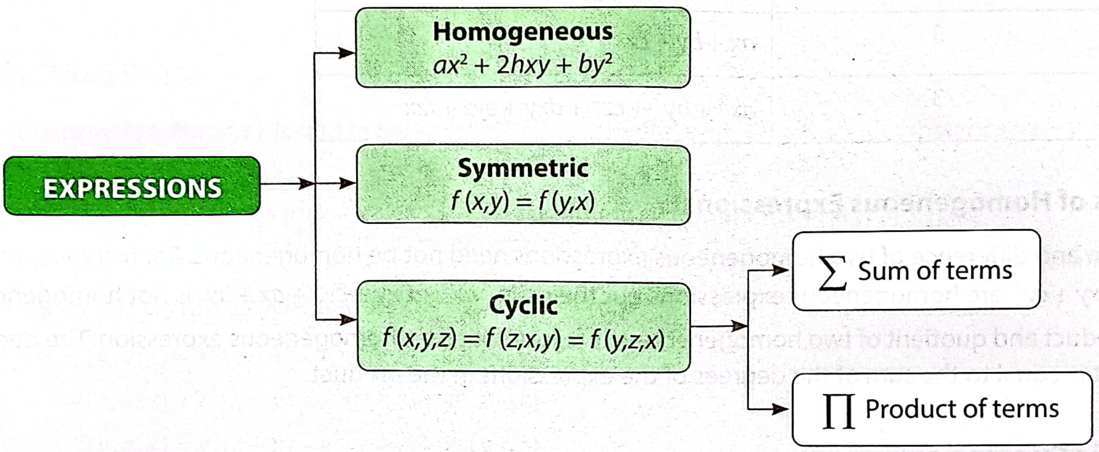
19. The remainder is ________ when $2x^2-5x-1$ is divided by $x-3$.
20. ________ is a factor of x^2-3x-4.
21. The remainder obtained when $3x^2-5x+6$ is divided by $x-2$ is ________.
22. If –4 is a root of $x^4+ax^3-19x^2-46x+120$, then the value of *a* is ________.

V. TRUE OR FALSE

23. The remainder is 425 when $5x^2+3x-7$ is divided by $x+9$.
24. The remainder obtained when $2x^2-3x+5$ is divided by $2x-1$ is 4.
25. The remainder obtained when $3x^2+6x-4$ is divided by $5x+2$ is –5.92.
26. $x-3$ is a factor of $2x^3-x^2-21x+18$.
27. $x-1$ is a factor of $6x^3+5x^2-2x-1$.

3.06 Cyclic Expressions

MIND MAP



INTRODUCTION

Recall that an algebraic expression is a combination of variables and constants using one or more mathematical operations. So far, we have studied about operations on algebraic expressions in general and their classification into monomials, binomials, trinomials, etc. In this chapter, we will study about another classification of algebraic expressions, based on the type of their terms and the nature of the expression when its variables are interchanged.

Homogeneous Expression

A homogeneous expression is an algebraic expression in which all the terms are of the same degree.

Examples:

(i) Consider $ax^2+2hxy+by^2$. Observe that

$$ax^2 \to \text{degree } 2$$
$$2hxy \to \text{degree } 2$$
$$by^2 \to \text{degree } 2$$

Therefore, $ax^2+2hxy+by^2$ is a homogeneous expression.

(ii) In $ax^4+bx^3y+cx^2y$, observe that

$$ax^4 \to \text{degree } 4$$
$$bx^3y \to \text{degree } 4$$
$$cx^2y \to \text{degree } \boxed{3}$$

So, $ax^2+2hxy+by^2$ is not a homogeneous expression.

A homogeneous expression is said to be complete if it contains all the possible terms in it.

For instance, $ax^2 + by^2$ is a homogeneous expression but it is not complete as the *xy* term is missing here, whereas $ax^3 + bx^2y + cxy^2 + dy^3$ is a complete homogeneous expression.

The standard form of some of the trivial complete homogeneous expressions is given in the following table:

Degree	Number of Variables	General Form
1	2	$ax + by$
2	2	$ax^2 + bxy + cy^2$
1	3	$ax + by + cz$
2	3	$ax^2 + by^2 + cz^2 + dxy + eyz + fzx$

Properties of Homogeneous Expressions

(i) The sum and difference of two homogeneous expressions need not be homogeneous. For instance, $ax + by$ and $ax^2 + bxy + cy^2$ are homogeneous expressions but their sum $ax^2 + bxy + cy^2 + ax + by$ is not homogeneous.

(ii) The product and quotient of two homogeneous expressions is also a homogeneous expression. The degree of the product is equal to the sum of the degrees of the expressions in the product.

Symmetric Expression

An algebraic expression in two variables is said to be symmetric if its value remains unchanged when the variables are interchanged.

For instance, if *x* and *y* are interchanged in $ax + b + cy$, the value of the expression remains unchanged. Hence, it is a symmetric expression. Similarly, $ax^2 + bxy + ay^2 + cx + cy + d$ is also a symmetric expression.

In general, an expression $f(x,y)$ is said to be symmetric if $f(x,y) = f(y,x)$. In case we have an expression in three variables, say $f(x,y,z)$, then it is said to be symmetric in *x* and *y* if $f(x,y,z) = f(y,x,z)$.

Key point

An expression $f(x,y,z)$ is said to be absolutely symmetric if $f(x,y,z)$ is symmetric in *x*-*y*, *y*-*z*, and *z*-*x*.

Example: Is $f(x,y,z) = ax^2 + 2bxyz + ay^2 + cz$ symmetric in *x* and *y*? Is $f(x,y,z)$ absolutely symmetric?

Solution: Observe that $f(y,x,z) = ay^2 + 2byxz + ax^2 + cz = f(x,y,z)$. So, $f(x,y,z)$ is symmetric in *x* and *y*.

Let us now check if $f(x,y,z)$ is symmetric in *y* and *z*.

$$f(x,z,y) = ax^2 + 2bxzy + az^2 + cy \neq f(x,y,z)$$

Therefore, $f(x,y,z)$ is not absolutely symmetric.

Key point

The sum, difference, product, and quotient of two symmetric expressions is also a symmetric expression.

Properties of Symmetric Expressions

To check how symmetric expressions behave under mathematical operations, let us consider an example. Let $3x+3y$ and $2x+2y$ be two symmetric expressions.

(i) $3x+3y+2x+2y=5x+5y$ (Sum is symmetric)

(ii) $(3x+3y)-(2x+2y)=x+y$ (Difference is symmetric)

(iii) $(3x+3y)(2x+2y)=6x^2+12xy+6y^2$ (Product is symmetric)

(iv) $\frac{6x^2+12xy+6y^2}{3x+3y}=2x+2y$ (Quotient is symmetric)

CYCLIC EXPRESSION

An algebraic expression $f(x,y,z)$ is said to be a cyclic if $f(x,y,z)=f(z,x,y)=f(y,z,x)$. For instance, $x+y+z$ is a cyclic expression.

Example: Is $f(x,y,z)=x^3(y-z)+y^3(z-x)+z^3(x-y)$ absolutely symmetric? Is $f(x,y,z)$ cyclic?

Solution: Observe that $f(y,x,z)=y^3(x-z)+x^3(z-y)+z^3(y-x)\neq f(x,y,z)$. So, $f(x,y,z)$ is not absolutely symmetric. Let us now check if $f(x,y,z)$ is cyclic or not.

$$f(x,y,z)=x^3(y-z)+y^3(z-x)+z^3(x-y)$$
$$f(z,x,y)=z^3(x-y)+x^3(y-z)+y^3(z-x)$$
$$f(y,z,x)=y^3(z-x)+z^3(x-y)+x^3(y-z)$$

Since $f(x,y,z)=f(z,x,y)=f(y,z,x)$, $f(x,y,z)$ is a cyclic expression.

For ease of representation, the symbols Σ (read as sigma) and Π (read as capital pi) are used to write cyclic expressions.

(i) Sigma: Σ is a Greek alphabet, used mainly to represent sum of terms.

(ii) Capital Pi: Π is also a Greek alphabet, used mainly to represent product of terms.

For instance, if a cyclic expression is given as $x(y+z)+y(z+x)+z(x+y)$, then it can be written as $\sum_{x,y,z} x(y+z)$, that is, $\sum_{x,y,z} x(y+z)=x(y+z)+y(z+x)+z(x+y)$.

Similarly, if a cyclic expression is given as $(a^2+b^2)(b^2+c^2)(c^2+a^2)$, then it can be written as $\prod_{a,b,c}(a^2+b^2)$, that is, $\prod_{a,b,c}(a^2+b^2)=(a^2+b^2)(b^2+c^2)(c^2+a^2)$.

Examples

(i) $\sum_{a,b,c}(a+1)(a+2)(b-c)=(a+1)(a+2)(b-c)+(b+1)(b+2)(c-a)+(c+1)(c+2)(a-b)$

(ii) $\sum_{a,b,c}(a+1)^3(b^2-c^2)=(a+1)^3(b^2-c^2)+(b+1)^3(c^2-a^2)+(c+1)^3(a^2-b^2)$

(iii) $\prod_{a,b,c}(a+b+c)=(a+b+c)(c+a+b)(b+c+a)=(a+b+c)^3$

(iv) $\prod_{a,b,c}a(b+c)=abc(b+c)(c+a)(a+b)$

Key point

The factors of a cyclic expression are also cyclic, that is, if $x-y$ is a factor of a cyclic expression $f(x,y,z)$, then its other two factors will be $y-z$ and $z-x$.

Factorization of a Cyclic Expression

Factorization of cyclic expressions makes use of the factor theorem, as illustrated in the following examples:

Example: Factorize $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

Solution: Let us assume that $a^2(b-c)+b^2(c-a)+c^2(a-b)$ is an expression in variable a. Then, we can write $f(a)=a^2(b-c)+b^2(c-a)+c^2(a-b)$.

Put $a=b$ and simplify.

$$f(b)=b^2(b-c)+b^2(c-b)+c^2(b-b)=0$$

Therefore, by the factor theorem, $a-b$ is a factor of $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

Observe that $a^2(b-c)+b^2(c-a)+c^2(a-b)$ is a cyclic expression and so, its factors will also be cyclic. Hence, the other two factors are $b-c$ and $c-a$.

The final step in this factorization is determining the constant *k* such that

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=k(a-b)(b-c)(c-a)$$

Since the above equation is an equality, we can assign specific values to *a*, *b*, and *c* to determine the value of *k*.

Let $a=0$, $b=1$, and $c=2$.

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=k(a-b)(b-c)(c-a)$$

$$0^2(1-2)+1^2(2-0)+2^2(0-1)=k(0-1)(1-2)(2-0)$$

$$2-4=2k$$

$$k=-1$$

$\therefore$ $$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a).$$

Example: Factorize $a^3(b-c)+b^3(c-a)+c^2(a-b)$.

Solution: Let us assume that $a^3(b-c)+b^3(c-a)+c^2(a-b)$ is an expression in variable a. Then, we can write $f(a)=a^3(b-c)+b^3(c-a)+c^2(a-b)$.

Put $a=b$ and simplify.

$$f(b)=b^3(b-c)+b^3(c-b)+c^2(b-b)=0$$

Therefore, by the factor theorem, $a-b$ is a factor of $a^3(b-c)+b^3(c-a)+c^2(a-b)$.

Observe that $a^3(b-c)+b^3(c-a)+c^2(a-b)$ is a cyclic expression and so, its factors will also be cyclic. Hence, the other two factors are $b-c$ and $c-a$.

Since the product $(a-b)(b-c)(c-a)$ is of degree three and $a^3(b-c)+b^3(c-a)+c^2(a-b)$ is a degree four expression, the linear cyclic factor $(a+b+c)$ must be multiplied with $(a-b)(b-c)(c-a)$.

The final step in this factorization is determining the constant *k* such that

$$a^3(b-c)+b^3(c-a)+c^2(a-b)=k(a-b)(b-c)(c-a)(a+b+c)$$

Since the above equation is an equality, we can assign specific values to *a*, *b*, and *c* to determine the value of *k*.

Let $a=0$, $b=-1$, and $c=2$.

$$a^3(b-c)+b^3(c-a)+c^2(a-b)=k(a-b)(b-c)(c-a)(a+b+c)$$
$$0^2(-1-2)+(-1)^2(2-0)+2^2(0+1)=k(0+1)(-1-2)(2-0)(0-1+2)$$
$$2+4=-6k$$
$$k=-1$$

$\therefore$ $a^3(b-c)+b^3(c-a)+c^2(a-b)=-1(a-b)(b-c)(c-a)(a+b+c)$.

Key point

Principle of Indeterminate Coefficients: It states that if two polynomials in one variable having the same degree are equal, then the coefficients of like powers in the two polynomials are individually equal.

For instance, if $ax^2+bx+c=px^2+qx+r$, then $a=p$, $b=q$, and $c=r$.

MISCONCEPTION

$$\prod_{x,y,z}(x^2-y^2)=(x^2-y^2)+(y^2-z^2)+(z^2-x^2)$$

FACT: $\prod_{x,y,z}(x^2-y^2)=(x^2-y^2)(y^2-z^2)(z^2-x^2)$

EXAMPLES

1. Differentiate between the symbols Σ and Π used to write cyclic expressions.

Solution: Σ and Π are both Greek alphabets. Σ is used to represent sum of terms, whereas Π is used to represent product of terms in a cyclic expression.

2. Expand $\sum_{a,b}a^2b$, $\sum_{a,b,c}a^2b$, and $\sum_{a,b,c,d}a^2b$.

Solution: $\sum_{a,b}a^2b=a^2b+b^2a$

$$\sum_{a,b,c}a^2b=a^2b+b^2c+c^2a$$

$$\sum_{a,b,c,d}a^2b=a^2b+b^2c+c^2d+d^2a$$

3. Expand $\left(\sum_{a,b,c}a\right)^2$ and $\sum_{a,b,c}a^2(b-c)$.

Solution: $\left(\sum_{a,b,c}a\right)^2=(a+b+c)^2$

$$\sum_{a,b,c}a^2(b-c)=a^2(b-c)+b^2(c-a)+c^2(a-b)$$

4. Expand $\prod_{a,b,c}(a^2-b^2)$.

Solution: $\prod_{a,b,c}(a^2-b^2)=(a^2-b^2)(b^2-c^2)(c^2-a^2)$

5. Expand $\prod_{a,b,c}(x+c)$.

Solution $\prod_{a,b,c}(x+c)=(x+a)(x+b)(x+c)$

RECALL

1. If all the terms of an algebraic expresion are of the same degree, then it is known as a homogeneous expression.
2. A homogeneous expression is said to be complete if it contains all the possible terms in it.
3. An expression $f(x,y)$ is said to be symmetric if $f(x,y)=f(y,x)$ and if $f(x,y,z)=f(y,x,z)$, then $f(x,y,z)$ is said to be symmetric in *x* and *y*.
4. If $f(x,y,z)$ is symmetric in *x* and *y*, *y* and *z*, and *z* and *x*, then $f(x,y,z)$ is said to be absolutely symmetric.
5. An expression $f(x,y,z)$ is said to be cyclic if $f(x,y,z)=f(y,z,x)=f(z,x,y)$.
6. Σ (sigma) is a Greek alphabet which is used to represent sum of terms.
7. Π (capital pi) is a Greek alphabet which is used to represent product of terms.

DESCRIPTIVE QUESTIONS

I. VERY SHORT ANSWER QUESTIONS

1. What are homogeneous expressions?
2. Give two examples of homogeneous expressions.
3. Is px^2+qy^2 homogeneous? If so, is it complete?
4. Is $px^3+qx^2y+rxy^2+sy^3$ homogeneous? If so, is it complete?
5. What is a symmetric expression?
6. Expand: $\left[\sum_{a,b,c}a\right]^3-\left[\sum_{a,b,c}a^3\right]$
7. Simplify: $\left[\sum_{a,b,c}a\right]^4-\prod_{a,b,c}(a+b)^4+\sum_{a,b,c}a^4$
8. Simplify: $\sum_{a,b,c}(a^2+b^2)(a-b)^2$
9. Simplify: $\sum_{a,b,c}(a+b)(a+2)(b-c)$
10. Simplify: $\prod_{a,b,c}\left[(a+1)^3+b^2+c\right]$

II. SHORT ANSWER QUESTIONS

11. What is a cyclic expression? Explain it.
12. Simplify: $\prod_{a,b,c}(a-b)(a+b)$
13. Simplify: $\sum_{a,b,c}(b^2-c^2)(b^2+c^2+bc)$
14. Simplify: $\prod_{x,y,z}x^2(y^2-z^2)$
15. Simplify: $\left(\sum_{a,b,c}a\right)^2-\left(\sum_{a,b,c}a^2\right)$

III. LONG ANSWER QUESTIONS

16. List the properties of a homogeneous expression.
17. Simplify: $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$
18. Factorize: $(a+b+c)^5-(a^5+b^5+c^5)$

IV. FILL IN THE BLANKS

19. $\sum_{a,b,c} a\left(b^2 - c^2\right) =$ ________

20. $\prod_{a,b,c}\left(a^2 - b^2\right) =$ ________

21. $\left(\sum_{a,b,c} a\right)^3 =$ ________

22. Product of two homogeneous expressions is ________.

23. Expression $ax + ay$ is ________ and ________.

24. The product and the quotient of two homogeneous expressions are ________ expressions.

25. The sum, difference, product, and quotient of two symmetric expressions are ________.

26. The degree of a cyclic expression helps determine the number of ________ it has.

27. $5x^2 + 13xyz + 12xy - 9y^2$ is not ________ as its degree of terms is 2, 3, 2, and 2 (i.e., not equal).

28. In the homogeneous expression $ax^4 + bx^3y + cx^2y$, ________ are represented by the coefficients *a*, *b*, and *c*, and ________ are represented by *x* and *y*.

29. An expression that remains same after interchanging the terms $a+b$, $b+c$, and $c+a$ is called a cyclic expression and its factors are ________ in nature.

V. TRUE OR FALSE

30. $ax^2 + 2bxy + by^2$ is said to be a second-degree homogeneous expression.

31. An expression $f(x, y)$ is said to be symmetric if $f(x, y) \neq f(y, x)$.

32. An algebraic expression $f(x, y, z)$ in three variable x, y, z is said to be cyclic if $f(x,y,z) = f(y,z,x) = f(z,x,y)$.

33. $\prod_{a,b,c}(x+a) = (x+a)(x+b)(x+c)$

34. $\left(\sum_{a,b,c} a\right)^3 = a^3 + b^3 + c^3$

35. $3x^2 + 5xy + 9z^2$ is a homogeneous expression where degree of each term is equal to 2.

36. A homogeneous expression is said to be complete if it contains all the possible terms in it.

3 | Algebra

PRACTICE SHEET

SINGLE CORRECT QUESTIONS

1. Simplify $5(3t-4)-(t^2+2)-2t(t-3)$.

(A) $-3t^2-21t-22$ (B) $3t^2+21t+22$ (C) $-3t^2+21t-22$ (D) $3t^2-21t+22$

2. The roots of the equation $4x^2+4a^2+(a^4-b^4)=0$ are

(A) $\frac{1}{2}(-a^2-b^2)$ (B) $\frac{1}{2}(a^2+b^2)$ (C) $\frac{1}{2}(a^2-b^2)$ (D) Both (B) and (C)

3. The simplified form of the expression $\left(\frac{-2r}{s}\right)\left(\frac{s^2}{-6t}\right)$ is

(A) $\frac{rs}{3t}$ (B) $\frac{rst}{2}$ (C) $\frac{rs}{4t}$ (D) $\frac{rs}{2t}$

4. If α and β are the root of the equation $3x^2-2x-8=0$, then $\alpha^2-\alpha\beta+\beta^2=$

(A) $\frac{76}{9}$ (B) $\frac{25}{3}$ (C) $\frac{16}{3}$ (D) $\frac{32}{3}$

5. The nature of roots of the quadratic equation $3p^2-2p+5=0$ is

(A) real and equal (B) real and unequal (C) complex (D) rational

6. The value of q for which the equation $(q+1)x^2-2(q-1)x+1=0$ has equal roots is

(A) 0 (B) 3 (C) either (A) or (B) (D) 1

7. If α and β are the roots of $ax^2+bx+c=0$, then $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=$

(A) $\frac{b^2-2ac}{a^2}$ (B) $\frac{-b}{c}$ (C) $\frac{b^2-2ac}{c^2}$ (D) $\frac{b^2-2ac}{ac}$

8. If α and β are the roots of $4x^2+3x+7=0$, then the value of $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ is

(A) $\frac{-27}{64}$ (B) $\frac{63}{16}$ (C) $\frac{225}{243}$ (D) $\frac{225}{243}$

9. If $2x^3+ax^2-5x+6=0$ is exactly divisible by $x-3$, then the value of a is

(A) 75 (B) −5 (C) −15 (D) 15

10. One of the factors of $x^3 - x^2 + 4x - 4$ is

(A) $x - 1$ (B) $x + 1$ (C) $x - 2$ (D) $x + 4$

11. If $kx^3 + 9x^2 + 4x - 8$ leaves a remainder -20 when divided by $x + 3$, then the value of k is

(A) 3 (B) 2 (C) 4 (D) 5

12. The remainder obtained when $x^3 + 3x^2 + 3x + 1$ is divided by $x - \frac{1}{2}$ is

(A) 0 (B) 1 (C) $\frac{27}{8}$ (D) $-\frac{26}{8}$

13. Expand $\sum_{a,b} a^2 b$.

(A) $a^2b + b^2a$ (B) $b^2c + c^2b$ (C) b^2a (D) $a^2b + b^2a + ab$

14. Expand $\sum_{a,b,c} (a^2 + b^2)(a - b)^2$.

(A) $(a^2 + b^2)(a - b)^2 + (b^2 + c^2)(b - c)^2$

(B) $(a^2 + b^2)(a - b)^2 + (b^2 - c^2)(b + c)^2 + (c^2 + a^2)(c - a)^2$

(C) $(a^2 + b^2)(a - b)^2 + (b^2 + c^2)(b - c)^2 + (c^2 + a^2)(c - a)^2$

(D) $(a^2 + b^2)(a - b)^2 + (c^2 + a^2)(c - a)^2$

15. Simplify $\frac{x}{x^2 + x - 2} - \frac{2}{x^2 - 5x + 4}$.

(A) $\frac{x^2 + 6x + 4}{(x-1)(x+2)(x+4)}$ (B) $\frac{x^2 - 6x - 4}{(x+1)(x-2)(x-4)}$ (C) $\frac{x^2 - 6x - 4}{(x-1)(x+2)(x-4)}$ (D) $\frac{x^2 + 6x + 4}{(x-1)(x+2)(x-4)}$

SINGLE CORRECT QUESTIONS

16. The square root of $x^{m^2 - n^2} \cdot x^{n^2 + 2mn} \cdot x^{n^2}$ is

(A) x^{m+n} (B) $x^{(m+n)^2}$ (C) $x^{\frac{m+n}{2}}$ (D) $x^{\frac{1}{2}(m+n)^2}$

17. Simplify $\frac{x^{-1} + y^{-1}}{(x + y)^{-1}}$.

(A) $\frac{(y - x)^2}{xy}$ (B) $\frac{(x - y)^2}{xy}$ (C) $\frac{(x + y)}{xy}$ (D) $\frac{(y + x)^2}{xy}$

18. If the equation $(1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0$ has equal roots, then

(A) $c^2 = (1 + m^2)$ (B) $a^2 = c^2(1 + m^2)$ (C) $c^2 = a^2(1 + m^2)$ (D) $c^2 = a^2(m^2)$

19. If $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ are the roots of a quadratic equation such that $\alpha + \beta = -7$ and $\alpha\beta = 12$, then the equation is

(A) $3x^2 + 50x + 49 = 0$ (B) $x^2 + 25x + 50 = 0$ (C) $x^2 - 50x + 49 = 0$ (D) $-7x^2 + 12x + 49 = 0$

20. If α and β are roots of $ax^2+bx+c=0$, then the equation whose roots are $2+\alpha$ and $2+\beta$ is

(A) $ax^2+(4a-b)x+4a+c-2b=0$ **(B)** $ax^2+(4a-b)x+4a+c+2b=0$

(C) $ax^2+(b-4a)x+4a+c+2b=0$ **(D)** $ax^2-(b-4a)x+4a+c-2b=0$

21. If $2x^3+3x^2+ax+b$ is divided by $x+2$, it leaves a remainder 2 and if divided by $x-2$, it leaves a remainder -2. The values of a and b, respectively, are

(A) 7, 162 **(B)** −7, −16 **(C)** 9, −16 **(D)** −9, −12

22. If the polynomials ax^3+4x^2+3x-4 and x^3-4x+a leave the same remainder when divided by $x-3$, then the value of a is

(A) 1 **(B)** 2 **(C)** −1 **(D)** −3

23. Expand $\sum\limits_{x,y,z} x^2\left(y^2-z^2\right)$.

(A) 0 **(B)** 1 **(C)** 2 **(D)** Not defined

24. Expand $\sum\limits_{a,b,c} a^2b$.

(A) $\sum\limits_{a,b,c} a^2b=a^2b+b^2a$ **(B)** $\sum\limits_{a,b,c} a^2b=b^2c+c^2a$

(C) $\sum\limits_{a,b,c} a^2b=a^2b+b^2c+c^2a$ **(D)** $\sum\limits_{a,b,c} a^2b=a^2b+c^2a$

25. Expand $\sum\limits_{a,b,c} (b-c)(b+c)$.

(A) 0 **(B)** 1 **(C)** −1 **(D)** Not defined

SINGLE CORRECT QUESTIONS

26. The unique solution to the system $30x-6y=3,\ -10x+5y=4$ is

(A) $\frac{30}{13}, \frac{3}{5}$ **(B)** $\frac{3}{5}, \frac{30}{13}$ **(C)** $\frac{13}{30}, \frac{5}{3}$ **(D)** $\frac{3}{13}, \frac{1}{6}$

27. If α and β are the roots of the equation $x^2-2x+3=0$, then the equation whose roots are $\frac{\alpha-1}{\alpha+1}$ and $\frac{\beta-1}{\beta+1}$ is

(A) $3x^2-2x-1=0$ **(B)** $x^2-3x+1=0$ **(C)** $3x^2-2x+1=0$ **(D)** $3x^2+2x+1=0$

28. Given that α and β are the roots of the equation $ax^2+bx+c=0$, the equation whose roots are $\alpha^2+\beta^2, \frac{1}{\alpha^2}+\frac{1}{\beta^2}$ is

(A) $a^2c^2x^2+\left(b^2-2ac\right)\left(a^2+c^2\right)x+\left(b^2-2ac\right)^2=0$ **(B)** $a^2c^2x^2-\left(b^2-2ac\right)\left(a^2+c^2\right)x+\left(b^2-2ac\right)^2=0$

(C) $a^2c^2x^2-\left(b^2-2ac\right)\left(a^2+c^2\right)x-\left(b^2-2ac\right)^2=0$ **(D)** $a^2c^2x^2+\left(b^2-2ac\right)\left(a^2+c^2\right)x-\left(b^2-2ac\right)^2=0$

29. If α and β are the roots of the equation $8x^2-3x+27=0$, then the value of $\left(\frac{\alpha^2}{\beta}\right)^{\frac{1}{3}}+\left(\frac{\beta^2}{\alpha}\right)^{\frac{1}{3}}$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{7}{2}$ (D) 4

30. The expanded form of $\prod_{a,b,c}(x+c)$ is

(A) $(2a)(2b)(2c)$ (B) $(a+b)(b+c)(c+a)$ (C) $(x+a)(x+b)(x+c)$ (D) abc

Read the two statements carefully to choose the correct option out of the options given below:

(A) Both statements are true.
(B) Both statements are false.
(C) Statement I is true. Statement II is false.
(D) Statement I is false. Statement II is true.

STATEMENT BASED QUESTIONS

31. Statement I: $t^3+1=(t+1)(t^2-t+1)$

Statement II: $4t^2-12t+9=(2t+3)^2$

32. Statement I: Simplified form of $\dfrac{1+\frac{1}{c-1}}{1-\frac{1}{c-1}}$ is $\dfrac{c}{c-2}$.

Statement II: Simplified form of $1+\dfrac{1}{1+\frac{1}{1+x}}$ is $\dfrac{3-2x}{2-x}$.

STATEMENT BASED QUESTIONS

33. Statement I: The number of real roots of $3^{2x^2-7x+7}=9$ is 2.

Statement II: If $\Delta<0$, a quadratic equation has real and unequal roots.

34. Statement I: If α,β are the roots of $ax^2+bx+c=0$, then $\left[\dfrac{1}{\alpha^2-\alpha\beta+\beta^2}\right]\left[\dfrac{\alpha}{\beta^2}+\dfrac{\beta}{\alpha^2}\right]=\dfrac{ab}{c^2}$.

Statement II: If α,β are the roots of $ax^2+bx+c=0$, then $\alpha+\beta=\dfrac{-b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

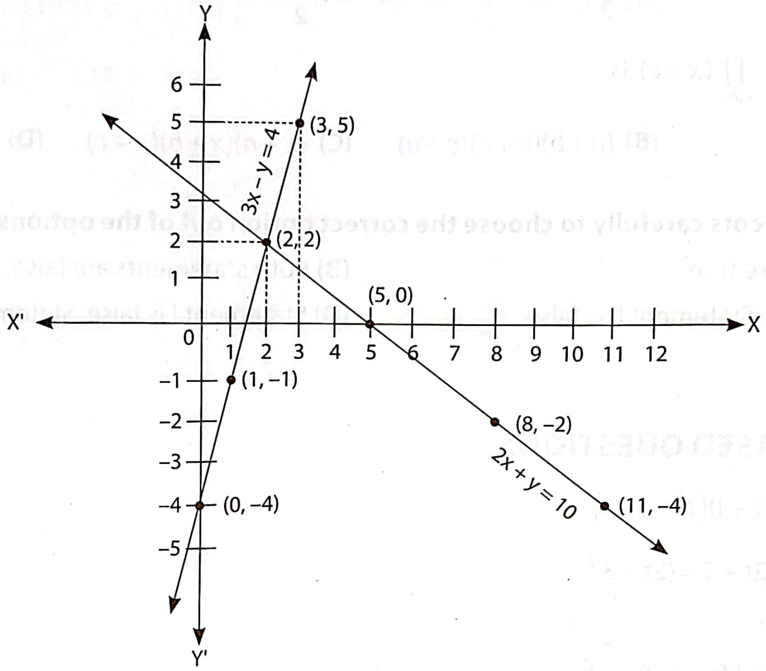
STATEMENT BASED QUESTION

35. Statement I: The expanded form of $\sum_{a,b,c}(a+1)^3\left(a^2-c^2\right)$ is $(a+1)^3\left(b^2-c^2\right)+(b+1)^3\left(c^2-a^2\right)+(b+1)^3\left(c^2-a^2\right)$.

Statement II: Π is a Greek alphabet which is mainly used to find the sum of given things or terms.

MULTI CORRECT QUESTIONS

36. The given graph illustrates the solution of two equations. Since this solution can also be a solution to another system of equations, identify the equations from the options below:



(A) $4x+6y=20$ **(B)** $4x+3y=20$ **(C)** $6x+2y=8$ **(D)** $6x-2y=8$

37. If α,β are the roots $x^2+x+1=0$, then

(A) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=1$ **(B)** $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=1$ **(C)** $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=-1$ **(D)** $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=-1$

MULTI CORRECT QUESTIONS

38. Find the value of a if the division of x^3+5x^2-ax+6 by $x-1$ leaves a remainder $2a$.

(A) $\frac{12}{3}$ **(B)** 4 **(C)** 12 **(D)** 16

39. Which of the following divisor-remainder combination is correct when the dividend is $2x^3-3x^2+4x-5$.

(A) $x+3,-98$ **(B)** $x-2,7$ **(C)** $x+1,-14$ **(D)** $x-1,2$

MULTI CORRECT QUESTIONS

40. Identify the correct expanded form.

(A) $\sum_{a,b}(a+b)=(a+b)+(b+c)+(c+a)$ **(B)** $\prod_{a,b,c}(a+b)=(a+b)(b+c)(c+a)$

(C) $\sum_{a,b,c}a=abc$ **(D)** $\prod_{a,b,c}\left(a^2+b^2\right)=\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)$

INTEGER TYPE QUESTION

41. Find the value of α if $\frac{x^9(2x)^4}{x^3}=\alpha x^{10}$.

INTEGER TYPE QUESTION

42. If α and β are the roots of the equation $x^2-2x-1=0$, then the value $|\alpha^2\beta+\alpha\beta^2|$ is _____.

COMPREHENSION TYPE QUESTIONS

If a function $p(x)=ax^3+bx^2+cx+d$ is divided by $x-k$, then to determine the value of the remainder, put $x-k=0$ which gives $x=k$. On substituting k for x in $p(x)$, we get the remainder of division. This is the remainder theorem.

43. Find the remainder when x^3+3x^2+3x+1 is divided by $x+1$.

(A) 1 **(B)** 4 **(C)** 0 **(D)** 5

44. Find the remainder when x^3+3x^2+3x+1 is divided by $x-\frac{1}{2}$.

(A) -3 **(B)** $\frac{27}{8}$ **(C)** $\frac{11}{8}$ **(D)** 1

45. If the remainder obtained on dividing x^3+3x^2+3x+1 by $x+k$ is -1, then the value of k is

(A) 2 **(B)** -1 **(C)** 0 **(D)** -2

COMPREHENSION TYPE QUESTION

If α,β are the roots of $ax^2+bx+c=0$, then $\alpha+\beta=\frac{-b}{\alpha}$ and $\alpha\beta=\frac{c}{a}$.

46. If α,β are the roots of $2x^2-3x+5=0$, then $\alpha^2+\beta^2=$

(A) $\frac{5}{2}$ **(B)** $\frac{11}{2}$ **(C)** $\frac{-11}{2}$ **(D)** $\frac{-11}{4}$

COMPREHENSION TYPE QUESTIONS

47. If α,β are the roots of $ax^2+bx+c=0$, then $\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2}=$

(A) $\frac{3abc-b^3}{a^3}$ **(B)** $\frac{3abc-b^3}{a^2c}$ **(C)** $\frac{3abc-b^3}{ac^2}$ **(D)** $\frac{3abc-b^3}{c^3}$

48. If α,β are the roots of $ax^2+bx+c=0$, then $\frac{\alpha^{-3}}{\beta^3}+\frac{\beta^{-3}}{\alpha^3}=$

(A) $\frac{2c^3}{a^3}$ **(B)** $\frac{2a^3}{c^3}$ **(C)** $\frac{2b^3}{a^3}$ **(D)** $\frac{2a^3}{b^3}$

MATRIX MATCH QUESTION

49. Match the real roots of the equations in column I with those given in column II.

Note: $|x| =$ modulus of x

Column I	Column II
(a) $\lvert x\rvert^2 - 7\lvert x\rvert + 12 = 0$	**(i)** ± 4
(b) $\lvert x\rvert^2 + 7\lvert x\rvert + 10 = 0$	**(ii)** ± 5
(c) $\lvert x\rvert^2 - \lvert x\rvert - 6 = 0$	**(iii)** ± 2
(d) $\lvert x\rvert^2 - 25 = 0$	**(iv)** ± 3
	(v) no real roots

(A) a-i,iv; b-v; c-iv; d-ii **(B)** a-i,iii; b-ii; c-v; d-iv **(C)** a-v; b-i,ii; c-i; d-iii **(D)** a-v; b-iv; c-ii; d-v

MATRIX MATCH QUESTION

50. Column II gives the value of the polynomials in column I at the mentioned points. Match them correctly.

Column I	Column II
(a) $3t^3 - 2t^2 + t$ at $t = \frac{1}{2}$	**(i)** 2
(b) $3x^3 + x^2 - 20x + 12$ at $x = \frac{2}{3}$	**(ii)** -1
(c) $7u^5 - 4u^3 + 2$ at $u = 0$	**(iii)** $\frac{3}{8}$
(d) $(3s-1)(2s-3)$ at $s = \frac{1}{2}$	**(iv)** 0
	(v) $\frac{2}{3}$

(A) a-v; b-iii; c-ii; d-i **(B)** a-iii; b-iv; c-i; d-ii **(C)** a-i; b-iv; c-iii; d-iv **(D)** a-ii; b-i; c-v; d-ii

KEY BOX

3. ALGEBRA

Single Correct Questions

1	2	3	4	5	6	7	8	9	10
C	A	A	A	C	C	C	C	B	A
11	12	13	14	15	16	17	18	19	20
A	C	A	C	C	A	D	C	C	D
21	22	23	24	25	26	27	28	29	30
D	C	A	C	A	C	C	B	A	C

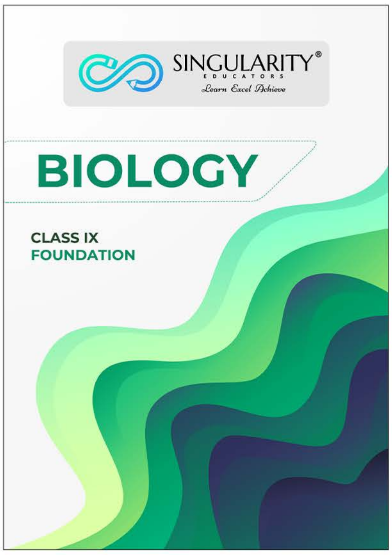
Statement Based Questions									
31	32	33	34	35					
C	C	C	D	B					

Multi Correct Questions				
36	37	38	39	40
A, D	C, D	A, B	A, B, C	B, D

Integer Type Questions									
41	42								
16	2								

Comprehension Type Questions									
43	44	45	46	47	48				
C	B	D	D	C	B				

Matrix Match Questions	
49	50
A	B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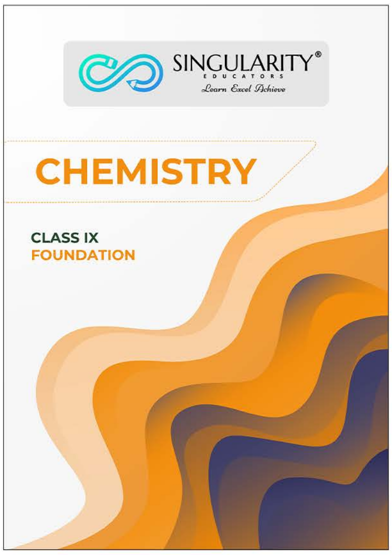
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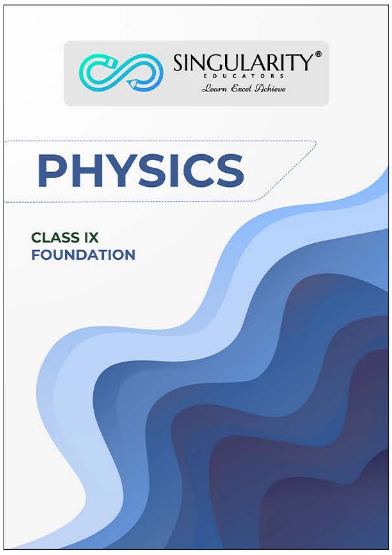
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